Name: MAS 3105 (Linear Algebra)
Test 1, Friday May 22, 2015

Remember that you won’t get any credit if you do not show the steps to your answers. You may show your work on the back of each page. There are 15 bonus points which do not carry over to other assignments or exams.

1. [20] a) Find the reduced row echelon form of the matrix $B = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \end{pmatrix}$.

b) Find the null space $N(B)$ of $B$, and specify a spanning set for $N(B)$.

$B = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \end{pmatrix} \xrightarrow{\text{RREF of } B} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

$N(B) = \{ x \in \mathbb{R}^3 \mid Bx = 0 \}$ is equivalent to $Ux = 0$, where $U = \text{RREF of } B$. So $x_2 + 2x_3 + 3x_4 = 0 \rightarrow x_2 = -2x_3 - 3x_4$, and $x_1 - x_3 - 2x_4 = 0 \rightarrow x_1 = x_3 + 2x_4$. Hence $N(B) = \{ (x_1, x_3 + 3x_2, 2x_3) \}$, and $N(B)$ is a spanning set for $N(B)$.

2. [20] State whether each of the following statement is true or false. No explanation needed.

a) If $A$ and $B$ are $n \times n$ matrices, then $(A - B)(A + B) = A^2 - B^2$. **False**, $AB \neq BA$ in general.

b) If $S$ is a nonempty subset of a vector space $V$, then $S$ contains the null vector of $V$. **False**

c) If $H$ is a $11 \times 17$ matrix, then $H^T$ is a $17 \times 11$ matrix. **True**, by defn of transpose.

d) If $A$ is a singular $22 \times 22$ matrix, then $A^T$ is singular. **True**, by Theorem 2.1.2 and Theorem 2.2.2.

e) If $U$, and $W$ are subspaces of $\mathbb{R}^n$, then $U \cup W$ is a subspace of $\mathbb{R}^n$. **False**, see example solved in class.

f) If $A$ and $B$ are $n \times n$ singular matrices, then $A + B$ is also singular. **False**

g) If $A = (a_1, a_2, a_3)$ is a $4 \times 3$ matrix with $N(A) = \{0\}$, and $b = -3a_1 + 4a_2 - 7a_3$, then $Ax = b$ has a unique solution. **True**

h) If $A$ and $U$ are $n \times n$ matrices and $U$ is nonsingular, then $\det(U^{-1}AU) = \det(A)$. **True**, by Th 2.2.3 and $\det(U^{-1}) = \frac{1}{\det(U)}$.

i) If $A$ is a $6 \times 5$ matrix and $B$ is a $5 \times 6$ matrix, then the product $AB$ is a $6 \times 6$ matrix. **True**, by defn of product.

j) If $A$ is a $12 \times 12$ matrix and $x = 0$ is the only solution to $Ax = 0$, then $A$ is nonsingular. **True**, by Th 1.5.2.

Let $n \geq 2$, let $p < n$.

Set $A = E_{11} + \cdots + E_{pp}$, $B = E_{pp+1} + \cdots + E_{nn}$, then $A + B = I_n$, while both $A$ and $B$ are singular. Remember $E_{ij}$ = matrix with 1 as entry at $i$th row and $j$th column, and 0 as entry everywhere else.
3. [20] Find the inverse of the matrix \( A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{pmatrix} \).

\[
\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}
\]

Hence \( A^{-1} = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 3 & -2 \\ 1 & 1 & 1 \end{pmatrix} \).

4. [25] a) Let \( A \) be a nonsingular \( n \times n \). Show that \( A^T A \) is nonsingular and \( \det(A^T A) > 0 \).

\[
\det(A^T A) = \det(A)^2 \cdot \det(A^T) = \det(A)^2 \cdot \det(A)^2 = \det(A)^4 > 0
\]

So \( \det(A^T A) \neq 0 \) and \( A^T A \) is nonsingular with \( \det(A^T A) > 0 \).

b) Let \( A \) be an \( n \times n \) matrix with \( \det(A - 2I_n) = 0 \). Show that the system \( Ax = 22x \) has a nontrivial solution.

\[
Ax - 22x \text{ is equivalent to } (A - 22I_n)x = O \text{ on. Now } A - 22I_n
\]

is singular, by Theorem 2.2, as \( \det(A - 22I_n) = 0 \). Hence \( Ax = 22x \)

is a nontrivial solution by Theorem 1.5.2.

c) Let \( C = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \). Find the LU factorization of \( C \).

\[
\begin{pmatrix} -2 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} L \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}
\]

\( \det(C) = \det(LU) = \det(L) \cdot \det(U) = 1 \cdot (-2) = -2 \)

d) Compute \( \det(C) \) with \( C \) as in question c).

\[
\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}
\]

\( \det(C) = \det(LU) = \det(L) \cdot \det(U) = 1 \cdot (-2) = -2 \)

3. Method 2: Using \( \text{adj} A: A^{-1} = \text{adj} A / \det A \). \( \det A = (18-16) - (12-12) + (8-9) = 1 \)

\[
\text{adj} A = \begin{pmatrix} 3 & 4 & 2 \\ 4 & 6 & 3 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} = A^{-1}, \text{ as } \det(A) = 1
\]
5. [5] Let $E$ be a vector space, and let $U$ be a nonempty subset of $E$. Complete the sentence: $U$ is called a subspace of $E$ when

i) $x + y \in U$ whenever $x \in U$ and $y \in U$

ii) $\lambda x \in U$ whenever $x \in U$, and $\lambda$ is a scalar.

6. [5] Prove that any two nonsingular $n \times n$ matrices are row equivalent. Let $A, B$ be two nonsingular matrices. Then $A$ is row equivalent to $L_n$ and $B$ is row equivalent to $L_n$. So

$A = E_k \cdots E_1 I_n$ for some elementary matrices $E_1, \ldots, E_k$. Similarly

$B = E_k \cdots E_1 L_n$. So

$I_n = E_k^{-1} \cdots E_1^{-1} B$, and $A = E_k \cdots E_1 F_1^{-1} F_2^{-1} \cdots F_k^{-1} B$, hence $A$ and $B$ are row equivalent.

7. [10] Let $U$ and $V$ be subspaces of a vector space $E$. Set $W = \{z \in E; z = u + v$ with $u \in U$ and $v \in V\}$. Show that $W$ is a subspace of $E$. Since $u_1 \in U$, $u_2 \in V$, and $u_1 + u_2 \in W$, it follows that $u_1 + u_2 \in W$. Let $y \in W$, $z \in W$. Show $y + z \in W$. $y = u_1 + v_1$, $u_2 \in U$, $v_2 \in V$; so $y + z = u_1 + v_1 + u_2 + v_2 = (u_1 + u_2) + (v_1 + v_2) \in U + V$, $u_1 + u_2 \in U$ and $v_1 + v_2 \in V$; so $y + z \in W$. Let $\alpha$ be a scalar. Show $\alpha y \in W$.

$\alpha y = \alpha (u_1 + v_1) = \alpha u_1 + \alpha v_1 \in U + V$, since $\alpha u_1 \in U$ and $\alpha v_1 \in V$; so $\alpha y \in W$. Hence $W$ is a subspace of $E$.

8. [10] Use mathematical induction on $m$ to prove that for every integer $m \geq 2$, if $A_1, A_2, \ldots, A_m$ are $n \times n$ matrices, then one has $(A_1 A_2 \cdots A_m)^T = A_m^T \cdots A_2^T A_1^T$. 

**Basis Step:** $m = 2$. Show $(A_1 A_2)^T = A_2^T A_1^T$. $(A_1 A_2)^T = A_2^T A_1^T$ by transpose rule.

**Inductive Step.** Let $m \geq 2$. Suppose that the property holds for $m$, prove it for $m + 1$. In other words, show $(A_1 A_2 \cdots A_m A_{m+1})^T = A_m^T A_{m-1}^T \cdots A_2^T A_1^T$.

Now

$$(A_1 A_2 \cdots A_m A_{m+1})^T = ((A_1 A_2 \cdots A_m) A_{m+1})^T = A_m^T (A_1 A_2 \cdots A_m)^T.$$ 

By case $m = 2$

$$A_m^T A_1^T A_2^T.$$ 

By inductive hypothesis

Hence the property holds for all $m \geq 2$. 
