Remember that you won’t get any credit if you do not show the steps to your answers. You may show your work on the back of each page. Total: 105 points.

1. [20] Solve the linear system

\[
\begin{align*}
&x_2 + x_3 + x_4 = 0 \\
&3x_1 + 3x_2 - 4x_3 = 9 \\
&x_1 + x_2 + 2x_3 + x_4 = 6 \\
&2x_1 + 3x_2 + x_3 + 3x_4 = 6
\end{align*}
\]

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
3 & 3 & -4 & 0 \\
1 & 1 & 2 & 1 \\
2 & 3 & 1 & 3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
9 \\
6 \\
6
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 2 & 1 & 6 \\
0 & -10 & -3 & -9 \\
0 & 1 & 1 & 6 \\
0 & -4 & 0 & -6
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 2 & 1 & 6 \\
0 & -4 & 0 & -6 \\
0 & -10 & -3 & -9
\end{pmatrix}
\]

\[
S = \left\{ \left( \frac{9}{2}, \frac{1}{2}, \frac{3}{2}, -2 \right) \right\}
\]
2. [20] State whether each of the following statement is true or false. No explanations needed.

(1) If $A$ is an $n \times n$ matrix with $A^2 = 0_M_n$, then $A$ is singular. True, $\det(A^2) = 0 = (\det A)^2$, so $\det A = 0$.

(2) If $A$ is a $20 \times 20$ matrix that is row equivalent to a nonsingular matrix $B$, then $\det(A) \neq 0$. True, $A = E_1 E_2 \cdots E_k B$, $E_i$ are elementary matrices.

(3) If $A$ is an $11 \times 15$ matrix, then $A^T A$ is a $15 \times 15$ matrix. True, by the definition of $A$ and product multiplication.

(4) If $U$ is a nonempty subset of a vector space $E$, then $U$ is a subspace of $E$. False, $U$ must be closed under addition and scalar multiplication.

(5) If $A$ and $B$ satisfy $\det(A) = \det(B)$, then $\det(AB) \geq 0$. True, $\det(AB) = \det(A) \det(B)$, under addition and scalar multiplication.

(6) If $A^2 - 3A + I_n = 0_M_n$, then $A$ is nonsingular. True.

(7) If $A$ is a $15 \times 15$ matrix, then $A^T$ is also nonsingular. False, pick $A = 0_{15 \times 15}$.

(8) If $A$ and $B$ are $n \times n$ matrices, then $\det(AB) = \det(BA)$. True, $\det(AB) = \det(A) \det(B)$.

(9) If $A$ and $B$ are nonsingular $n \times n$ matrices, then $A + B$ is also nonsingular. False, pick $A = I_n$, $B = -I_n$.

(10) If $A$ and $B$ are $15 \times 15$ matrices with $A = B^T$, then $\det(A) = \det(B)$. True, $\det(A) = \det(B^T) = \det(B)$.

3. [15] Consider the linear system whose augmented matrix is

$$
\begin{bmatrix}
1 & 2 & 1 & 0 \\
2 & m & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 9-2m & 0
\end{bmatrix}
$$

a) Is it possible for this system to be inconsistent? Explain, or no credit.

This system cannot be inconsistent since $x = 0_{15}$ is a solution.

b) For which value(s) on $m$ will the system have infinitely many solutions? Write down the solution set(s) in this case.

We have

$$
(9-2m)x_3 = 0 \iff m \neq 9/2,
$$

If $m \neq 9/2$, then $x_2 + 2x_3 = 0 \Rightarrow x_2 = -2x_3$ and $x_1 + 2x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3 = x_3 - 3x_3 = -2x_3$. Thus, $x_3$ is arbitrary. So

$$
S = \{ (x_1, x_2, x_3) \mid x_1 = -x_2 - x_3, x_2 = -2x_3 \}
$$

So the system has infinitely many solutions when $m = 9/2$.

4. [10] For which values of the number $a$ do we have $A_a^2 = I_2$ if $A_a = \begin{bmatrix} a-1 & 1 \\ -2 & 1-a \end{bmatrix}$, and $I_2$ denotes the identity matrix of order 2?

$$
A_a^2 = \begin{pmatrix}
(a-1)^2-2 & a-1+1-a \\
-2(a-1)-2(1-a) & (a-1)^2-2
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iff \begin{pmatrix}
(a-1)^2-2 \\
-2(a-1)-2(1-a)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

$(a-1)^2 - 2 = 1 \rightarrow (a-1)^2 = 3 \rightarrow a = 1 \pm \sqrt{3}$. 

5. [20] Find the inverse of the matrix (Hint. You may use the reduction method.)

\[ A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 1 & 3 & 5 \end{pmatrix} \]

We start with the augmented matrix \((A | I_3)\):

\[
\begin{pmatrix}
1 & 1 & 1 & | & 1 & 0 & 0 \\
2 & 3 & 5 & | & 0 & 1 & 0 \\
1 & 3 & 5 & | & 0 & 0 & 1
\end{pmatrix}
\]

Reduce to:

\[
\begin{pmatrix}
1 & 1 & 1 & | & 1 & 0 & 0 \\
0 & -5 & -4 & | & -2 & 1 & 0 \\
0 & 0 & -2 & | & -1 & 0 & 1
\end{pmatrix}
\]

Hence, \( A^{-1} = \begin{pmatrix} \frac{5}{2} & -2 & \frac{3}{2} \\ -2 & 1 & -1 \end{pmatrix} \).

6. [10] a) Let \( A \) be an \( n \times n \) matrix. Set \( B = A + A^T \) and \( D = A^T - A \). Show that

\( B \) is symmetric and \( D \) is skew symmetric.

\[
B^T = (A + A^T)^T = A^T + A = A^T = B; \ 	ext{so} \ B \ 	ext{is symmetric}
\]

\[
D^T = (A^T - A)^T = A^T - A = A - A^T = -(A^T - A) = -D; \ 	ext{so} \ D \ 	ext{is skew symmetric}
\]

b) Let \( A \) denote the matrix in problem 5. Find an upper triangular matrix \( U \) and a lower triangular matrix \( L \) such that \( A = LU \).

From Pb 5, \( U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \), \( L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \)

\[
LU = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 1 & 3 & 5 \end{pmatrix}, \ 	ext{by definition of product, so} \ A = LU
\]

7. [10] a) Let \( V \) be a vector space, and let \( S \) be a subset of \( V \). Complete the sentence: \( S \) is called a subspace of \( V \) when

i) \( Ax \in S \) for all scalar \( a \) and all \( x \) in \( S \)

ii) \( x + y \in S \) for all \( x, y \) in \( S \).

b) Let \( A \) and \( B \) be \( m \times m \) matrices with \( AB = A + B \). Show that, if \( B \) is nonsingular, then \( A \) is nonsingular.

\[
\text{If} \ B \ 	ext{is nonsingular, then} \ B^{-1} \ 	ext{exists, so}
\]

\[
(AB)B^{-1} = (A + B)B^{-1} = AB^{-1} + BB^{-1} = AB^{-1} + I
\]

\[
A = AB^{-1} + I; \ 	ext{hence} \ A - AB^{-1} = I \ 	ext{or}
\]

\[
A(Im - B^{-1}) = I; \ 	ext{so} \ A \ 	ext{is nonsingular.}
\]