BESSEL EQUATIONS AND BESSEL FUNCTIONS

Bessel functions form a class of the so called *special functions*. They are important in math as well as in physical sciences (physics and engineering). They are especially important in solving boundary values problems in cylindrical coordinates. First we define another important function: *the Gamma function* which is used in the series expansion of the Bessel functions, then we construct the Bessel functions J_α and Y_α.

1. The Gamma Function

The *Gamma* function (also called *Euler's integral*) is the function defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty \mathrm{e}^{-s} s^{x-1} ds \ .$$

The improper integral defining Γ is convergent for $x > 0$. To see why, note that for every $x > 0$,

$$\lim_{s \to \infty} \frac{\mathrm{e}^{-s} s^{x-1}}{s^{-2}} = 0 \ .$$

Thus there exists $M > 0$ such that $\mathrm{e}^{-s} s^{x-1} \leq s^{-2}$ for $s > M$. This implies that

$$\int_M^\infty \mathrm{e}^{-s} s^{x-1} ds \leq \int_M^\infty \frac{ds}{s^2} = \frac{1}{M} \ .$$

Also for $s \in (0,\ M)$, $\mathrm{e}^{-s} s^{x-1} \leq s^{x-1}$ and

$$\int_0^M \mathrm{e}^{-s} s^{x-1} ds \leq \int_0^M s^{x-1} ds = \left[\frac{s^x}{x} \right]_{s=0}^{s=M} = \frac{M^x}{x} \ .$$

We have then

$$\int_0^\infty \mathrm{e}^{-s} s^{x-1} ds = \int_M^\infty \mathrm{e}^{-s} s^{x-1} ds + \int_0^M \mathrm{e}^{-s} s^{x-1} ds \leq \frac{1}{M} + \frac{M^x}{x} \ .$$

This shows that $\Gamma(x)$ is well defined for $x > 0$.

The most important property of the Gamma function is given in the following lemma.

Lemma 1. *The function Γ satisfies the following*

$$\Gamma(x+1) = x\Gamma(x) \,, \qquad \forall x > 0 \ .$$

Proof. The proof is simply an integration by parts

$$\int_0^A \mathrm{e}^{-s} s^x ds = \left[-\mathrm{e}^{-s} s^x \right]_0^A + x \int_0^A \mathrm{e}^{-s} s^{x-1} ds = x \int_0^A \mathrm{e}^{-s} s^{x-1} ds \ - \ A^x \mathrm{e}^{-A} \ .$$

By taking the limit as $A \to \infty$, we get $\Gamma(x+1) = x\Gamma(x)$

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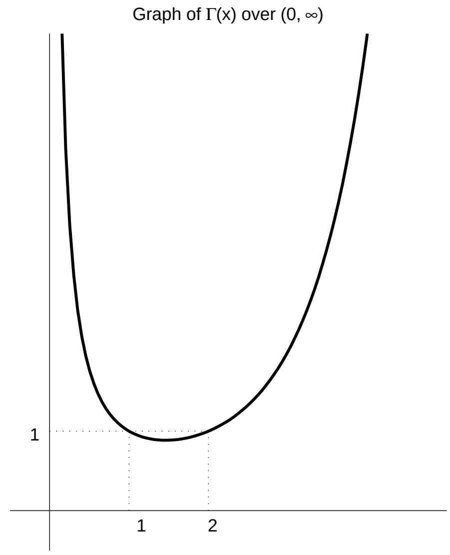
It can be shown that Γ has derivatives of all orders for $x > 0$ and that Γ has a unique extremum (global minimum) on the interval $(0,\ \infty)$. The minimum is reached at a number $x_0 \in (1,\ 2)$ and $\Gamma(x_0) < 1$. Furthermore, Γ satisfies

$$\lim_{x\to 0^+} \Gamma(x) = \infty, \quad \text{and} \quad \lim_{x\to\infty} \Gamma(x) = \infty$$



We can use the fundamental property to extend Γ as a smooth functions to $\mathbb{R}\backslash\{0, -1, -2, \cdots\}$ (the whole real line except 0 and the negative integers). First we extend Γ to the interval $(-1,\ 0)$ by defining

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \qquad \text{for} \quad x \in (-1,\ 0)$$

(note the above definition makes sense since $x+1 \in (0,\ 1)$ and $\Gamma(x+1)$ is defined by the integral). Once, Γ is defined on $(-1,\ 0)$, we extend it to the interval $(-2,\ -1)$ by using the same property. More precisely, if Γ is defined on the interval $(-j,\ -(j-1))$ with $j \in \mathbb{Z}^+$, then we extend it to the interval $(-(j+1),\ -j)$ by using the fundamental property. We have in particular that $\lim_{x\to k} |\Gamma(x)| = \infty$ for $k = 0,\ -1,\ -2,\ \cdots$

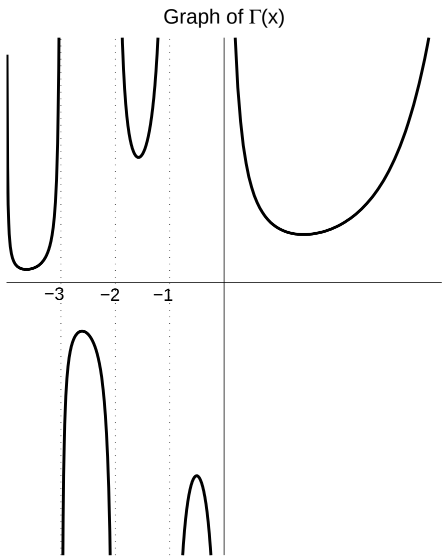
Now we compute some values of the Gamma function.

$$\Gamma(1) = \int_0^\infty \mathrm{e}^{-s} ds = 1\ .$$

By using the fundamental property of Γ, we get easily its values at the positive integers.

$$\begin{aligned}
&\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1 \\
&\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2 = 2! \\
&\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3 \cdot 2! = 3! \\
&\vdots \\
&\Gamma(n+1) = n! \qquad \forall n \in \mathbb{Z}^+
\end{aligned}$$

The Gamma function appears as an interpolation of the factorial function.



To compute $\Gamma(1/2)$ we use the value of the Gaussian integral $\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}\,/2$ (you have probably encountered this integral in Multivariable Calculus (MAC2313) or in Prob./Statistics class). In the following calculation, we have made the substitution $t = \sqrt{s}$.

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-s} s^{-1/2} ds = 2\int_0^\infty e^{-t^2} dt = \sqrt{\pi}\ .$$

The Gamma function satisfies many other identities such:

Reflection formula : $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin \pi x} \quad (x \neq 0, \pm 1, \pm 2, \cdots)$

Duplication formula : $\Gamma(2x) = \dfrac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x + \dfrac{1}{2}\right) \quad (2x \neq 0,\ -1,\ -2, \cdots)$

2. Bessel's Equation

Bessel's equation of order α (with $\alpha \geq 0$) is the second order differential equation

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0 \tag{1}$$

In order to find all solutions we need two independent solutions. We are going to construct the independent solutions for $x > 0$.

2.1. **Construction of a first solution.** Note that $x = 0$ is a singular point of the equation. More precisely, it is a *regular singular point* (see your notes from the first differential equations class, MAP2302). For such differential equations, we can use the *method of Fröbenius* to construct series solutions. We seek a (formal) series solution

$$y = x^r \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^{k+r} \tag{2}$$

of equation (1), with $c_k \in \mathbb{R}$, and $c_0 \neq 0$. The substitution of this series and its (formal) derivatives into equation (1) gives

$$x^2 \sum_{k=0}^{\infty} (k+r)(k+r-1)c_k x^{k+r-2} + x \sum_{k=0}^{\infty} (k+r)c_k x^{k+r-1} + (x^2 - \alpha^2) \sum_{k=0}^{\infty} c_k x^{k+r} = 0$$

We rewrite this as

$$\sum_{k=0}^{\infty} (k+r)(k+r-1)c_k x^{k+r} + \sum_{k=0}^{\infty} (k+r)c_k x^{k+r} + \sum_{k=0}^{\infty} c_k x^{k+r+2} - \sum_{k=0}^{\infty} \alpha^2 c_k x^{k+r} = 0$$

then as

$$\sum_{k=0}^{\infty} (k+r)(k+r-1)c_k x^{k+r} + \sum_{k=0}^{\infty} (k+r)c_k x^{k+r} + \sum_{k=2}^{\infty} c_{k-2} x^{k+r} - \sum_{k=0}^{\infty} \alpha^2 c_k x^{k+r} = 0$$

After grouping the like terms and simplifying, we obtain

$$(r^2 - \alpha^2)c_0 x^r + ((r+1)^2 - \alpha^2)c_1 x^{r+1} + \sum_{k=2}^{\infty} \left[((r+k)^2 - \alpha^2)c_k + c_{k-2} \right] x^{k+r} = 0$$

In order for this series to be identically zero, each coefficient must be zero. We have then

$$\begin{aligned} &(r^2 - \alpha^2)c_0 = 0\,, \\ &\left((r+1)^2 - \alpha^2\right) c_1 = 0\,, \\ &\left((r+j)^2 - \alpha^2\right) c_j + c_{j-2} = 0\,, \qquad j = 2,\ 3,\ 4, \cdots \end{aligned}$$

Since $c_0 \neq 0$, then the first equation implies that r must satisfy

$$r^2 - \alpha^2 = 0\,.$$

This is the *indicial equation* of the Bessel equation. The indicial roots are

$$r = \alpha \quad \text{and} \quad r = -\alpha\,.$$

Consider the case $r = \alpha$. The second equation becomes

$$(2\alpha + 1)c_1 = 0 \quad \Rightarrow \quad c_1 = 0 \quad (\text{since } \alpha > 0)\,.$$

For $j \geq 2$ the recurrence relation becomes

$$\left((\alpha + j)^2 - \alpha^2\right) c_j + c_{j-2} = 0 \quad \Rightarrow \quad c_j = \frac{-c_{j-2}}{j(2\alpha + j)}\,.$$

Since $c_1 = 0$, the above relation gives

$$c_3 = \frac{-c_1}{3(2\alpha + 3)} = 0, \quad c_5 = \frac{-c_3}{5(2\alpha + 5)} = 0, \quad c_7 = 0, \quad \cdots$$

That is, all coefficients with odd indices are 0 ($c_{\text{odd}} = 0$). For the coefficients with even indices, we have

$$\begin{aligned} c_2 &= \frac{-c_0}{2(2\alpha + 2)} = \frac{-c_0}{4(1 + \alpha)} \\ c_4 &= \frac{-c_2}{4(2\alpha + 4)} = \frac{(-1)^2 c_0}{2^4 (2!)(1 + \alpha)(2 + \alpha)} \\ c_6 &= \frac{-c_4}{6(2\alpha + 6)} = \frac{(-1)^3 c_0}{2^6 (3!)(1 + \alpha)(2 + \alpha)(3 + \alpha)} \end{aligned}$$

A proof by induction gives

$$c_{2j} = \frac{(-1)^j c_0}{j! 2^{2j} (1 + \alpha)(2 + \alpha) \cdots (j + \alpha)}\,, \qquad j = 1,\ 2,\ 3,\ \cdots$$

A formal solution is therefore

$$y = \sum_{j=0}^{\infty} c_{2j} x^{2j+\alpha} = \sum_{j=0}^{\infty} \frac{(-1)^j c_0}{j! 2^{2j} (1+\alpha)(2+\alpha)\cdots(j+\alpha)} x^{2j+\alpha}$$

We are going to select c_0 and use the Gamma function to rewrite the series solution in a more compact form. It follows from the fundamental property of the Gamma function that

$$\begin{aligned}\Gamma(j+1+\alpha) \quad &= (j+\alpha)\Gamma(j+\alpha)\\ &= (j+\alpha)(j-1+\alpha)\Gamma(j-1+\alpha)\\ &\vdots\\ &= (j+\alpha)(j-1+\alpha)\cdots(1+\alpha)\Gamma(1+\alpha)\ .\end{aligned}$$

Equivalently,

$$(1+\alpha)(2+\alpha)\cdots(j+\alpha) = \frac{\Gamma(j+1+\alpha)}{\Gamma(1+\alpha)}\ .$$

We select c_0 as

$$c_0 = \frac{1}{2^\alpha \Gamma(1+\alpha)}\ .$$

With this choice of c_0, the particular series solution becomes

$$J_\alpha(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+1+\alpha)} \left(\frac{x}{2}\right)^{2j+\alpha}\ .$$

This solution is known as the *Bessel function of the first kind of order* α.

Now we determine the domain where the series converges. Note that

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+1+\alpha)} \left(\frac{x}{2}\right)^{2j}\ .$$

The last series is a power series in $(x/2)^2$. To find its radius of convergence, we can use the ratio test:

$$\lim_{j\to\infty} \left| \frac{(-1)^{j+1}/\left((1+j)!\Gamma(j+2+\alpha)\right)}{(-1)^j/\left(j!\Gamma(j+1+\alpha)\right)} \right| = \lim_{j\to\infty} \frac{1}{(j+1)(j+1+\alpha)} = 0\ .$$

The radius of convergence is infinite (the power series converges to an analytic function on $\mathbb{R}$). The function $J_\alpha(x)$ is defined for $x \geq 0$.

2.2. **Construction of a second solution.** Recall that the indicial roots of the Bessel equation are $r = \pm\alpha$. We have used $r = \alpha$ to construct the solution $J_\alpha(x)$. We can redo the above construction with $r = -\alpha$. However, this can be done only if $\alpha \notin \mathbb{Z}^+$. In this case a second independent solution of Bessel's equation is

$$J_{-\alpha}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+1-\alpha)} \left(\frac{x}{2}\right)^{2j-\alpha}\ .$$

Note that $J_{-\alpha}$ is not defined at $x = 0$. We have

$$\lim_{x\to 0^+} |J_{-\alpha}(x)| = \infty\ .$$

The general solution of equation in $(0,\ \infty)$ is

$$y(x) = AJ_\alpha(x) + BJ_{-\alpha}(x)\ ,$$

with A and B constants.

When $\alpha = n \in \mathbb{Z}^+$, the situation is a little more involved. The first solution is

$$J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+n)!} \left(\frac{x}{2}\right)^{2j+n} .$$

If we try to define J_{-n} by using the recurrence relations for the coefficients, then starting with $c_0 \neq 0$, we can get

$$\begin{aligned} c_2 &= \frac{-c_0}{2(2-2n)} = \frac{-c_0}{4(1-n)} \\ c_4 &= \frac{-c_2}{4(4-2n)} = \frac{(-1)^2 c_0}{2^4(2!)(1-n)(2-n)} \\ &\vdots \\ c_{2(n-1)} &= \frac{(-1)^{n-1} c_0}{(n-1)! 2^{2(n-1)}(1-n)(2-n)\cdots 2\cdot 1} \end{aligned}$$

At the order $2n$ however we get

$$0 c_{2n} - c_{2(n-1)} = 0 .$$

This is a contradiction since $c_{2(n-1)} \neq 0$. Thus, the recurrence relations will not lead to a series solution.

Another attempt to define J_{-n} is to define it as

$$J_{-n}(x) = \lim_{\alpha \to n} J_{-\alpha}(x).$$

In this case, we get back either $J_{-n} = \pm J_n$ and J_{-n} and J_n are dependent solutions of the equations. More precisely, we have the following lemma.

Lemma. *We have*

$$J_{-n}(x) = (-1)^n J_n(x)$$

Proof. For $\alpha \notin \mathbb{Z}^+$ (and α close to n), we have

$$J_{-\alpha}(x) = x^{-\alpha} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! 2^{2j-\alpha} \Gamma(j+1-\alpha)} x^{2j} .$$

Recall that $\lim_{z \to -p} |\Gamma(z)| = \infty$ for $p = 0$ or $p \in \mathbb{Z}^+$. When $\alpha \to n$, $(j+1-\alpha)$ tends to 0 or a negative integer for $j = 0, 1, 2, \cdots, (n-1)$. For such values of j, the coefficients of x^{2j} in the series above approaches 0:

$$\lim_{\alpha \to n} \frac{(-1)^j}{j! 2^{2j-\alpha} \Gamma(j+1-\alpha)} = 0 .$$

We get then,

$$J_{-n}(x) = \lim_{\alpha \to n} J_{-\alpha}(x) = x^{-n} \sum_{j=n}^{\infty} \frac{(-1)^j}{j! 2^{2j-n} \Gamma(j+1-n)} x^{2j} .$$

and after using the fundamental property of the Gamma function we obtain

$$J_{-n}(x) = \sum_{j=n}^{\infty} \frac{(-1)^j x^{2j-n}}{j! 2^{2j-n} (j-n)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+n} x^{2k+n}}{(k+n)! 2^{2k+n} k!} = (-1)^n J_n(x) .$$

Now we indicate how to construct a second independent solution of equation (1) when $\alpha = n$. Consider $\alpha = n + \epsilon$ with $0 < \epsilon < 1$ (hence such $\alpha \notin \mathbb{Z}^+$). The corresponding Bessel equation has two independent solutions $J_{n+\epsilon}$ and $J_{-(n+\epsilon)}$. The function $Y_{n+\epsilon}$ defined by

$$Y_{n+\epsilon}(x) = \frac{J_{n+\epsilon}(x) - (-1)^n J_{-(n+\epsilon)}(x)}{\epsilon} .$$

Since function $Y_{n+\epsilon}$ is a linear combination of $J_{n+\epsilon}$ and $J_{-(n+\epsilon)}$, then $Y_{n+\epsilon}$ is also a solution of the corresponding Bessel's equation of order $\alpha = n + \epsilon$. We define Y_n as:

$$Y_n(x) = \lim_{\epsilon \to 0} Y_{n+\epsilon}(x) = \lim_{\epsilon \to 0} \frac{J_{n+\epsilon}(x) - (-1)^n J_{-(n+\epsilon)}(x)}{\epsilon} .$$

It can be proved that the function Y_n is a solution of the Bessel equation of order n and that Y_n and J_n are independent (see for example R.Courant and D. Hilbert, *Method of Mathematical Physics*, vol. 2, or H. Sagan, *Boundary and Eigenvalue Problems of Mathematical Physics*). This solution Y_α is called the *Bessel function of the second kind of order n.* It can also be proved that

$$\lim_{x \to 0^+} Y_n(x) = -\infty .$$

Another method to obtain a second solution of the Bessel equation in the exceptional case is to seek it in the form

$$y(x) = J_n(x) \ln x + \sum_j C_j x^j .$$

The coefficients C_j are then found by a recurrence relation.

The explicit expression of the $Y_n(x)$ is given below. Its derivation can be found in advanced texts about special function. For $n \in Z^+$, we have

$$\begin{aligned} Y_n(x) &= \frac{2}{\pi} J_n(x) \left(\gamma + \ln \frac{x}{2} \right) - \frac{x^n}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j (c_{j+n} + c_j)}{2^{2j+n} (j!)(n+j)!} x^{2j} \\ &\qquad - \frac{1}{\pi x^n} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{2^{2j-n} (j!)} x^{2j} , \end{aligned}$$

where, for $j = 0,\ 1,\ 2,\ \cdots$, the constants c_j are given by

$$c_0 = 0,\ c_1 = 1,\ c_2 = 1 + \frac{1}{2}, \cdots , c_j = 1 + \frac{1}{2} + \cdots \frac{1}{j}$$

and where γ is the *Euler constant* given by

$$\gamma = \lim_{j \to \infty} \left(1 + \frac{1}{2} + \cdots \frac{1}{j} - \ln j \right) , \qquad \gamma \approx 0.57721....$$

For $n = 0$, we have

$$Y_0(x) = \frac{2}{\pi} J_n(x) \left(\gamma + \ln \frac{x}{2} \right) - \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j c_j}{2^{2j} (j!)^2} x^{2j}$$

2.3. **General solution of the Bessel equation.** We summarize the above discussions in the following theorem.

Theorem. *Given the Bessel equation of order $\alpha \geq 0$,*

$$x^2 y'' + xy' + (x^2 - \alpha^2) y = 0$$

then we have the followings:

- *If $\alpha \notin \mathbb{Z}^+ \cup \{0\}$, the equation has two independent solutions $J_\alpha(x)$ and $J_{-\alpha}(x)$ (Bessel functions of the first kind) and the general solution is*

$$y(x) = AJ_\alpha(x) + BJ_{-\alpha}(x) \ ,$$

where A and B are constants.
- *If $\alpha = n$ with $n = 0$ or $n \in \mathbb{Z}^+$, the equation has only one Bessel function of the first kind $J_n(x)$, another independent solution is the Bessel function of the second kind $Y_n(x)$. The general solution of the equation is*

$$y(x) = AJ_n(x) + BY_n(x) \ .$$

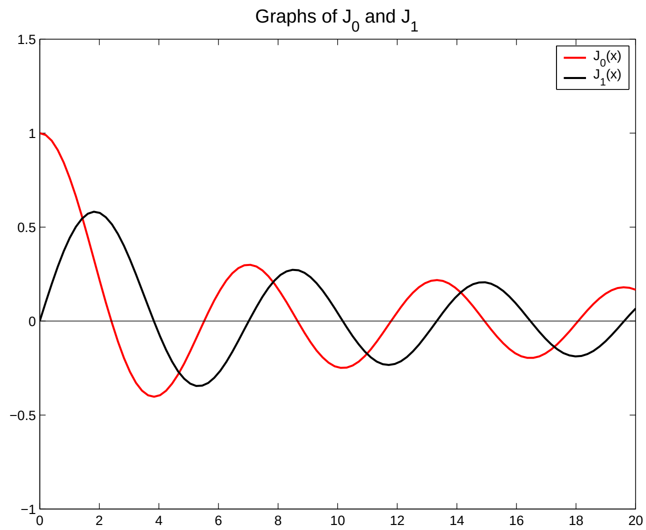
3. Remarks on Bessel functions

The expansions of the functions J_0 and J_1 are

$$J_0(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2} \left(\frac{x}{2}\right)^{2j} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots$$

$$J_1(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+1)!} \left(\frac{x}{2}\right)^{2j+1} = \frac{x}{2} - \frac{x^3}{2^3(2!)} + \frac{x^5}{2^5(2!)(3!)} - \frac{x^7}{2^7(3!)(4!)} + \cdots$$

The graphs of J_0 and of J_1 resemble those of cosine and sine with a decreasing



amplitude. Notice how the zeros of J_0 and J_1 behave. Between two consecutive zeros of J_0 there is exactly one zero of J_1. The following table lists the approximate values of the first 9 positive zeros of J_0 and J_1

j	1	2	3	4	5	6	7	8	9
J_0	2.405	5.520	8.654	11.792	14.931	18.071	21.212	24.353	27.494
J_1	3.832	7.016	10.174	13.324	16.471	19.616	22.760	25.904	29.047

For n large, the n-th zero of J_0 is approximately $n\pi - (\pi/4)$ and the n-th zero of J_1 is approximately $n\pi + (\pi/4)$. It is shown that for x large we have

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) \quad \text{and} \quad J_1(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{3\pi}{4}\right)$$

In fact the m-th Bessel function J_m has the following behavior

$$J_m(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{(2m+1)\pi}{4}\right) \quad \text{for } x \text{ large.}$$

This approximation shows that J_m has infinitely many positive zeros that tends to infinity. More precisely, we have following proposition about the zeros of Bessel functions.

Proposition 1. *For every $\alpha \in \mathbb{R}$, the positive zeros of J_α form an increasing unbounded sequence. That is, the solution set of the equation*

$$J_\alpha(x) = 0, \qquad x > 0\ ,$$

forms a sequence

$$0 < x_1 < x_2 < x_3 < \cdots < x_n < \cdots\ , \quad \text{with} \quad \lim_{n\to\infty} x_n = \infty$$

The proof of this proposition is beyond the aim of this course.

For $\alpha = \pm\frac{1}{2},\ \pm\frac{3}{2},\ \pm\frac{5}{2},\ \cdots, \pm\frac{2k+1}{2}, \cdots$, the Bessel functions J_α are elementary functions. This means that $J_\alpha(x)$ can be expressed algebraically in terms of $\sin x$, $\cos x$ and x. The following proposition gives the expressions of some Bessel functions with such indices.

Proposition 2. *We have the following relations*

$$\begin{aligned}
J_{1/2}(x) &= \sqrt{\frac{2}{\pi x}} \sin x\ ,\\
J_{-1/2}(x) &= \sqrt{\frac{2}{\pi x}} \cos x\ ,\\
J_{3/2}(x) &= \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right)\ ,\\
J_{-3/2}(x) &= -\sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x\right)\ .
\end{aligned}$$

Proof. We prove the first relation and leave the others as an exercise. Recall that the Taylor expansion of $\sin x$ is

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}\ .$$

We need the value of $\Gamma(j + (3/2))$. We have $\Gamma(1/2) = \sqrt{\pi}$ (see section about the Gamma function). By using the fundamental property of the Gamma function, we get

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}\,,$$
$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{(3\cdot 1)\sqrt{\pi}}{2^2}\,,$$
$$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{(5\cdot 3\cdot 1)\sqrt{\pi}}{2^3}\,.$$

We prove by induction that for $j \in \mathbb{Z}^+$,

$$\Gamma\left(j+\frac{3}{2}\right) = \frac{(2j+1)(2j-1)\cdots 3\cdot 1}{2^{j+1}}\sqrt{\pi}\,.$$

We can simplify the product of the odd integers above as

$$(2j+1)(2j-1)\cdots 5\cdot 3\cdot 1 = \frac{(2j+1)!}{(2j)(2j-2)\cdots 4\cdot 2} = \frac{(2j+1)!}{2^j(j!)}\,.$$

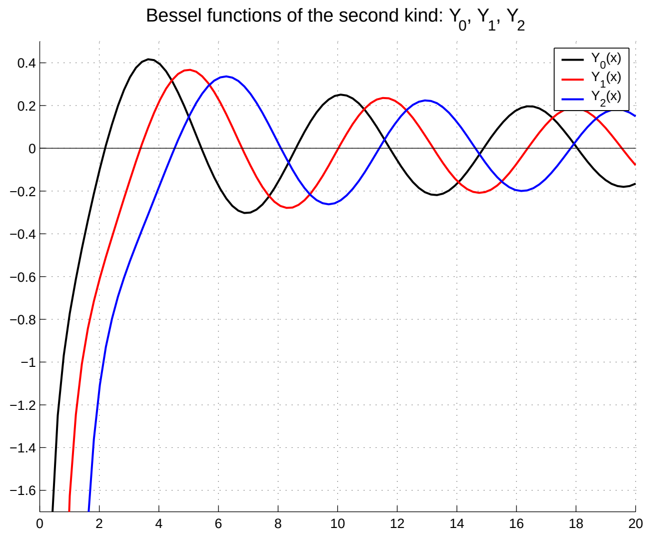
Hence,

$$\Gamma\left(j+\frac{3}{2}\right) = \frac{(2j+1)!}{2^{2j+1}(j!)}\sqrt{\pi}\,.$$

Now we use these to show the first relation of the proposition.

$$\begin{aligned}
J_{1/2}(x) &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+(3/2))}\left(\frac{x}{2}\right)^{2j+(1/2)} \\
&= \sqrt{\frac{2}{x}}\sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j+1}(j!)\Gamma(j+(3/2))}x^{2j+1} \\
&= \sqrt{\frac{2}{x}}\sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j+1}(j!)}{2^{2j+1}(j!)(2j+1)!\sqrt{\pi}}x^{2j+1} \\
&= \sqrt{\frac{2}{\pi x}}\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}x^{2j+1} \\
&= \sqrt{\frac{2}{\pi x}}\sin x
\end{aligned}$$

Analogous results about the behaviors of the Bessel functions of the second kind can be obtained.



4. Some properties of the Bessel functions J_α

The bessel functions satisfy a large number of properties. We limit ourself here to list the following.

Properties of J_α.

(1) $J_0(0) = 1$ and $J_\alpha(0) = 0$ if $\alpha > 0$.
(2) $J_n(x)$ is an even function if $n \in \mathbb{Z}^+$ is even and $J_n(x)$ is an odd function in n is odd.
(3) $J_{-n}(x) = (-1)^n J_n(x)$ for $n \in \mathbb{Z}^+$.
(4) $\dfrac{d}{dx}\big(x^{-\alpha} J_\alpha(x)\big) = -x^{-\alpha} J_{\alpha+1}(x)$.
(5) $\dfrac{d}{dx}(x^{\alpha} J_\alpha(x)) = x^{\alpha} J_{\alpha-1}(x)$.
(6) $\dfrac{d}{dx}(J_\alpha(x)) = \dfrac{1}{2}\left(J_{\alpha-1}(x) - J_{\alpha+1}(x)\right)$.
(7) $J_{\alpha+1}(x) + J_{\alpha-1}(x) = \dfrac{2\alpha}{x} J_\alpha(x)$.
(8) $\displaystyle\int x^{-\alpha} J_{\alpha+1}(x)dx = -x^{-\alpha} J_\alpha(x) + C$.
(9) $\displaystyle\int x^{\alpha} J_{\alpha-1}(x)dx = x^{\alpha} J_\alpha(x) + C$.

The first two properties are easy to obtain from the series representation of J_α and the third has already been verified.

Proof of 4. Multiply the series representation of J_α by $x^{-\alpha}$ and differentiate

$$\begin{aligned}\frac{d}{dx}\left(x^{-\alpha}J_\alpha(x)\right) &= \sum_{j=1}^{\infty}\frac{(-1)^j 2j}{j!\Gamma(j+\alpha+1)}\frac{x^{2j-1}}{2^{2j+\alpha}}\\ &= \sum_{j=1}^{\infty}\frac{(-1)^j}{(j-1)!\Gamma(j+\alpha+1)}\frac{x^{2j-1}}{2^{2j-1+\alpha}}\\ &= x^{-\alpha}\sum_{j=1}^{\infty}\frac{(-1)^j}{(j-1)!\Gamma(j+\alpha+1)}\left(\frac{x}{2}\right)^{2j-1+\alpha}\\ &= x^{-\alpha}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k!\Gamma(k+1+(\alpha+1))}\left(\frac{x}{2}\right)^{2k+(\alpha+1)}\\ &= -x^{-\alpha}J_{\alpha+1}(x)\end{aligned}$$

Proof of 5. Left as an exercise

Proof of 6. We have (take into account properties 4 and 5)

$$\begin{aligned}\frac{d}{dx}\left(J_\alpha(x)\right) &= \frac{d}{dx}\left(x^\alpha\left(x^{-\alpha}J_\alpha(x)\right)\right) = \alpha x^{\alpha-1}\left(x^{-\alpha}J_\alpha(x)\right) + x^\alpha\frac{d}{dx}\left(x^{-\alpha}J_\alpha(x)\right)\\ &= \alpha x^{-1}J_\alpha(x) + x^\alpha\left(-x^{-\alpha}J_{\alpha+1}(x)\right)\\ &= \alpha x^{-1}J_\alpha(x) - J_{\alpha+1}(x)\end{aligned}$$

Similarly

$$\begin{aligned}\frac{d}{dx}\left(J_\alpha(x)\right) &= \frac{d}{dx}\left(x^{-\alpha}\left(x^{\alpha}J_\alpha(x)\right)\right) = -\alpha x^{-\alpha-1}\left(x^{\alpha}J_\alpha(x)\right) + x^{-\alpha}\frac{d}{dx}\left(x^{\alpha}J_\alpha(x)\right)\\ &= -\alpha x^{-1}J_\alpha(x) + x^{-\alpha}\left(x^{\alpha}J_{\alpha-1}(x)\right)\\ &= -\alpha x^{-1}J_\alpha(x) + J_{\alpha-1}(x)\end{aligned}$$

By adding the two expressions we get

$$2\frac{d}{dx}\left(J_\alpha(x)\right) = J_{\alpha-1}(x) - J_{\alpha+1}(x)$$

Proof of 7. It follows from the proof of 6. that

$$\begin{aligned}\frac{d}{dx}\left(J_\alpha(x)\right) + \alpha x^{-1}J_\alpha(x) &= J_{\alpha-1}(x)\\ \frac{d}{dx}\left(J_\alpha(x)\right) - \alpha x^{-1}J_\alpha(x) &= -J_{\alpha+1}(x)\end{aligned}$$

We get, by subtraction,

$$2\alpha x^{-1}J_\alpha(x) = J_{\alpha-1}(x) + J_{\alpha+1}(x)$$

Proof of 8. It follows from property 4 that

$$\int x^{-\alpha}J_{\alpha+1}(x)dx = -\int\frac{d}{dx}\left(x^{-\alpha}J_\alpha(x)\right)dx = -x^{-\alpha}J_\alpha(x) + C$$

Proof of 9. Left as an exercise.

Example 1. We have proved in Proposition 2 that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$. In one of the exercises you will be asked to prove that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$ We can use property 7 with $\alpha = 1/2$ to deduce that

$$J_{3/2}(x) + J_{-1/2}(x) = \frac{1}{x} J_{1/2}(x)$$

Thus,

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right) .$$

Similar arguments can be used to prove that $J_{k+(1/2)}$ is an elementary function.

Example 2. We can use property 5 with $\alpha = 1$ to get

$$(xJ_1(x))' = xJ_0(x) \;\Leftrightarrow\; xJ_1'(x) + J_1(x) = xJ_0(x)$$

or

$$J_1'(x) = J_0(x) - \frac{J_1(x)}{x}$$

The following table lists the values $J_0(x)$ and of $J_1(x)$ for some values of x between 0 and 10.

x	0	0.5	1.0	1.5	2.0	2.5	3.0
$J_0(x)$	1.0000	0.9385	0.7652	0.5118	0.2239	-0.0484	-0.2601
$J_1(x)$	0	0.2423	0.4401	0.5579	0.5767	0.4971	0.3391
x	3.5	4	4.5	5	5.5	6	6.5
$J_0(x)$	-0.3801	-0.3971	-0.3205	-0.1776	-0.0068	0.1506	0.2601
$J_1(x)$	0.1374	-0.0660	-0.2311	-0.3276	-0.3414	-0.2767	-0.1538
x	7	7.5	8	8.5	9	9.5	10
$J_0(x)$	0.3001	0.2663	0.1717	0.0419	-0.0903	-0.1939	-0.2459
$J_1(x)$	-0.0047	0.1352	0.2346	0.2731	0.2453	0.1613	0.0435

By repeated use of property 7, we can get $J_n(x)$ for any integer n once $J_0(x)$ and $J_1(x)$ are known.

Example 3. Let use the table to find $J_4(3.5)$. We have $J_0(3.5) = -0.3801$ and $J_1(3.5) = 0.1374$. By using property 7 with $\alpha = 1$, then $\alpha = 2,\ 3$, and 4, we get

$$\begin{aligned} J_2(3.5) + J_0(3.5) &= \frac{2}{3.5} J_1(3.5) & \qquad J_2(3.5) &= 0.4586 \\ J_3(3.5) + J_1(3.5) &= \frac{4}{3.5} J_2(3.5) & \qquad J_3(3.5) &= 0.3868 \\ J_4(3.5) + J_2(3.5) &= \frac{6}{3.5} J_3(3.5) & \qquad J_4(3.5) &= 0.2044 \end{aligned}$$

Example 4. We use the integral property 8 and integration by parts to find the following integral

$$\begin{aligned} \int x^{-2} J_5(x)dx &= \int x^2 (x^{-4} J_5(x))dx \\ &= x^2(-x^{-4} J_4(x)) + \int (2x) x^{-4} J_4(x)dx \\ &= -x^{-2} J_4(x) + 2\int x^{-3} J_4(x)dx \\ &= -x^{-2} J_4(x) + 2x^{-3} J_3(x) + C \end{aligned}$$

5. An integral representation of $J_n(x)$

There is an interesting representation of the Bessel functions of the first kind with integer order n in terms of a definite integral. We have the following proposition.

Proposition 3. *For $n \in \mathbb{Z}$, we have*

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta \ .$$

Proof. Recall the Taylor expansion of the exponential function

$$\mathrm{e}^z = \sum_{j=0}^{\infty} \frac{z^j}{j!} \qquad \forall z \in \mathbb{C} \ .$$

(the series converges uniformly and absolutely for $|z| \leq R$ for every $R > 0$). We have then

$$\mathrm{e}^{xt/2} = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left(\frac{x}{2}\right)^j \quad \text{and} \quad \mathrm{e}^{-x/2t} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!t^j}\left(\frac{x}{2}\right)^j \ .$$

The product is

$$\mathrm{e}^{xt/2}\mathrm{e}^{-x/2t} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^j}{j!}\left(\frac{x}{2}\right)^j \frac{(-1)^k}{k!t^k}\left(\frac{x}{2}\right)^k = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{(j!)(k!)}\left(\frac{x}{2}\right)^{j+k} t^{j-k}$$

We rewrite this relation as a power series in t^n (so the coefficients will depend on x).

$$\mathrm{e}^{(x/2)(t-(1/t))} = \sum_{n=1}^{\infty} C_{-n}(x)\frac{1}{t^n} + C_0(x) + \sum_{n=1}^{\infty} C_n(x)t^n \ .$$

We need to show that $J_m(x) = C_m(x)$. The coefficient $C_n(x)$ is obtained from the double series by grouping all the coefficients of t^m. Thus all term with $j - k = m$:

$$C_m(x) = \sum_{j-k=m,\ j,k\geq 0} \frac{(-1)^k}{(j!)(k!)}\left(\frac{x}{2}\right)^{j+k}$$

or equivalently (by setting $k = j - m$),

$$C_m(x) = \sum_{j=0}^{\infty} \frac{(-1)^{j-m}}{(j!)(j-m)!}\left(\frac{x}{2}\right)^{2j-m} = (-1)^m \sum_{j=0}^{\infty} \frac{(-1)^{j-m}}{(j!)(j-m)!}\left(\frac{x}{2}\right)^{2j-m}$$

The last series is precisely $J_{-m}(x)$. We have then

$$C_m(x) = (-1)^m J_{-m}(x) = J_m(x) \ .$$

The expansion of $\mathrm{e}^{(x/2)(t-(1/t))}$ is therefore

$$\mathrm{e}^{(x/2)(t-(1/t))} = J_0(x) + \sum_{n=1}^{\infty} J_n(x)\left[t^n + \frac{(-1)^n}{t^n}\right] \ .$$

Now we evaluate the left side and the right side of the above expression for $t = \mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$. For $n \in \mathbb{Z}^+$, we have

$$t^n + \frac{(-1)^n}{t^n} = \mathrm{e}^{in\theta} + (-1)^n\mathrm{e}^{-in\theta} = \begin{cases} 2\cos(n\theta) & \text{if } n = 2p \text{ is even} \\ 2i\sin(n\theta) & \text{if } n = 2p+1 \text{ is odd} \end{cases}$$

and

$$e^{(x/2)(t-(1/t))} = e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta)\ .$$

It follows that

$$\cos(x\sin\theta)+i\sin(x\sin\theta) = J_0(x)+2\sum_{p=1}^{\infty}J_{2p}(x)\cos(2p\theta)+2i\sum_{p=0}^{\infty}J_{2p+1}(x)\sin(2p+1)\theta$$

By equating the real and imaginary parts, we get

$$\begin{aligned}\cos(x\sin\theta) &= J_0(x)+2\sum_{p=1}^{\infty}J_{2p}(x)\cos(2p\theta)\\ \sin(x\sin\theta) &= 2\sum_{p=0}^{\infty}J_{2p+1}(x)\sin(2p+1)\theta\end{aligned}$$

Recall the orthogonality of the trigonometric system

$$\frac{2}{\pi}\int_0^{\pi}\cos(j\theta)\cos(k\theta) = \frac{2}{\pi}\int_0^{\pi}\cos(j\theta)\cos(k\theta) = \begin{cases} 1 & \text{if } j=k\,,\\ 0 & \text{if } j\neq k\,.\end{cases}$$

By using these orthogonality relations and the above series, we get

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\cos(n\theta)d\theta &= \frac{2}{\pi}\int_0^{\pi}\left[J_0(x)+\sum_{p=1}^{\infty}J_{2p}(x)\cos(2p\theta)\right]\cos(n\theta)d\theta\\ &= \begin{cases} J_n(x) & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd.}\end{cases}\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta)\sin(n\theta)d\theta &= \frac{2}{\pi}\int_0^{\pi}\left[\sum_{p=0}^{\infty}J_{2p+1}(x)\sin(2p+1)\theta\right]\sin(n\theta)d\theta\\ &= \begin{cases} J_n(x) & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even.}\end{cases}\end{aligned}$$

By adding these relations we get for $n\in\mathbb{Z}$ that

$$\frac{1}{\pi}\int_0^{\pi}\left[\cos(x\sin\theta)\cos(n\theta)+\sin(x\sin\theta)\sin(n\theta)\right]d\theta = J_n(x)$$

which proves the proposition.

A immediate consequence of the integral representation is the following

Corollary. *For every $n\in\mathbb{Z}$, we have*

$$|J_n(x)|\leq 1, \qquad \forall x\in\mathbb{R}$$

and

$$\lim_{x\to\infty} J_n(x) = 0\ .$$

6. Exercises

Exercise 1. The table bellow lists approximate values of the Gamma function for values of x in the interval $[0,\ 1]$. Use the table together with the fundamental property of the Gamma function to find the following values

$$\Gamma(9.45),\ \ \Gamma(23.10),\ \ \Gamma(6.05),\ \ \Gamma(4.85),\ \ \Gamma(8.85),$$
$$\Gamma(-0.75),\ \ \Gamma(-4.65),\ \ \Gamma(-0.01),\ \ \Gamma(-2,85),\ \ \Gamma(-3.75).$$

x	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$\Gamma(x)$	19.470	9.513	6.220	4.591	3.626	2.992	2.546	2.218	1.968	1.773
x	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
$\Gamma(x)$	1.616	1.489	1.385	1.298	1.225	1.164	1.113	1.069	1.032	1.00

Exercise 2. The aim of this exercise is to establish the formulas

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta d\theta \qquad x > 0,\ y > 0 \qquad (*)$$

1. Show that

$$\Gamma(x)\Gamma(y) = \int_0^\infty \mathrm{e}^{-s} s^{x-1} ds \int_0^\infty \mathrm{e}^{-t} t^{y-1} dt = \int_0^\infty \int_0^\infty \mathrm{e}^{-(u^2+v^2)} u^{2x-1} v^{2y-1} du dv$$

(*Hint*: consider the substitutions $s = u^2$ and $t = v^2$)

2. Use polar coordinates $u = r\cos\theta$, $v = r\sin\theta$ to establish formula $(*)$.
3. Use formula $(*)$ to establish the following formula ($j, k \in \mathbb{Z}^+$)

$$\int_0^{\pi/2} \cos^{2j-1}\theta \cos^{2k-1}\theta d\theta = \frac{(j-1)!\,(k-1)!}{2\,(k+j-1)!}$$

4. Use formula $(*)$ together with $\Gamma(j + (1/2)) = \dfrac{(2j-1)!}{2^{2j-1}(j-1)!}\sqrt{\pi}$. to establish

$$\int_0^{\pi/2} \cos^{2j}\theta \cos^{2k-1}\theta d\theta = \frac{(2j-1)!(k-1)!(k+j-1)!}{2^{2j-1}(j-1)!(2k+2j-1)!}$$

(*Hint*: Use $x = j + (1/2)$ and $y = k$ in formula $(*)$.)

5. Use the table of values of the Gamma function given in exercise 1 to find an approximation of the integral

$$\int_0^{\pi/2} \cos^{\pi}\theta \sin^{\mathrm{e}}\theta d\theta$$

Exercise 3. The Psi function is defined as the logarithmic derivative of Γ:

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

Use the fundamental property of Γ to show that Ψ satisfies

$$\Psi(x+1) = \Psi(x) + \frac{1}{x}\ .$$

Exercise 4. Write the first five terms of the series representation of J_0; J_1; J_2; J_{-3}; $J_{3/4}$; $J_{1/5}$.

Exercise 5. Use the series expansion of $J_{-1/2}$ to establish

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \; .$$

You can also establish this formula by using property (5) with $\alpha = 1/2$ and $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

Exercise 6. Repeat the steps of example 1 to show that

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right) \; .$$

Exercise 7. Find the expressions of $J_{5/2}$ and of $J_{-5/2}$.

Exercise 8. Use the table of values of J_0 and J_1 to find the following values

$$J_2(.5), \quad J_3(5), \quad J_4(8.5)$$

Exercise 9. Prove that $\int_0^x sJ_0(s)ds = xJ_1(x)$.

Exercise 10. Find the integrals

$$\int x^9 J_8(x)dx, \quad \int x^{-3/2} J_{5/2}(x)dx, \qquad \int x^5 J_2(x)dx$$

Exercise 11. Find the integrals

$$\int x^{2-\alpha} J_{\alpha+1}(x)dx, \quad \int J_1(x)dx, \qquad \int J_2(x)dx$$

Exercise 12. Find the integrals

$$\int \left[J_3(x) - J_5(x)\right] dx, \quad \int_0^x s^4 J_1(s)ds$$

Exercise 13. Show that

$$\int_0^R x^\alpha J_{\alpha-1}(\lambda x)dx = \frac{R^\alpha}{\lambda} J_\alpha(\lambda R)$$

Exercise 14. Show that

$$x^2 J''_\alpha(x) - (\alpha^2 - \alpha - x^2) J_\alpha(x) - x J_{\alpha+1}(x) = 0$$

(*Hint*: Use Bessel's equation and property 4)

Exercise 15. Show that

$$\int_0^x J_3(s)ds = 1 - J_2(x) - 2\frac{J_1(x)}{x}$$

(*Hint*: Start with $J_3(s) = s^2(s^{-2}J_3(s))$ and use integration by parts)

Exercise 16. Use the expansion of $\cos(x\sin\theta)$ involved in the proof of Proposition 3 to show that

$$\cos x = J_0(x) + 2\sum_{j=1}^{\infty}(-1)^j J_{2j}(x)$$

$$\sin x = 2\sum_{j=0}^{\infty}(-1)^j J_{2j+1}(x)$$

$$1 = J_0(x) + 2\sum_{j=1}^{\infty} J_{2j}(x)$$

Exercise 17. Use the integral representation of $J_n(x)$ to show that

$$J_n'(x) = \frac{1}{\pi}\int_0^{\pi} \sin(n\theta - x\sin(\theta))\, \sin\theta \, d\theta$$