1 S-Matrix Formulas

A.1 Invariant Amplitude

The invariant amplitude $\mathcal{M}$ is related to the S-matrix through the relation

$$S_{\beta\alpha} = \delta_{\beta\alpha} - i(2\pi)^4\delta^4(p_\beta - p_\alpha)\frac{\mathcal{M}_{\beta\alpha}}{\sqrt{\prod_i (2E_i)}}, \tag{1}$$

where α and β label the initial and final states and the product of factors $(2E_i)$ is over both initial and final states.

A.2 Decay Processes $\alpha \to (1, 2, .., n)$.

The transition probability for the decay process $\alpha \to (1, 2, .., n)$ is given by

$$dW_{\beta\alpha} = \frac{(2\pi)^4}{2E_\alpha} \mid \mathcal{M}_{\beta\alpha} \mid^2 \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3(2E_i)}\delta^4(p_1 + p_2 + ... + p_n - p_\alpha). \tag{2}$$

For a two-body final state eq.(2) becomes:

$$dW_{\beta\alpha} = \frac{1}{32\pi^2} \mid \mathcal{M}_{\beta\alpha} \mid^2 \frac{p_{cm}d\Omega_{cm}}{M_\alpha^2} \tag{3}$$

where M_α is the parent mass and

$$p_{cm}^2 = \frac{1}{4M_\alpha^2}\left[M_\alpha^2 - (m_1 + m_2)^2\right]\left[M_\alpha^2 - (m_1 - m_2)^2\right] \tag{4}$$

A.3 2-to-2 Differential Cross section.

For an initial state $\alpha = (1, 2)$, where particle number 1 is the projectile and number 2 is the target, and a final state $\beta = (3, 4, .., n)$ the differential cross section is

$$d\sigma = \frac{(2\pi)^4 \mid \mathcal{M}_{\beta\alpha} \mid^2}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}} \prod_{i=3}^{n} \frac{d^3p_i}{(2\pi)^3(2E_i)}\delta^4\left(\sum_{i=3}^{n} p_i - p_\alpha\right). \tag{5}$$

The 2-to-2 collision differential cross section $(m_1 + m_2 \to m_3 + m_3)$ is given by

$$\frac{d\sigma}{d\Omega_{cm}}(s, \theta) = \frac{1}{64\pi^2 s}\frac{p'_{cm}}{p_{cm}} \mid \mathcal{M}_{\beta\alpha} \mid^2, \tag{6}$$

where $d\Omega = d(cos\theta_{cm})d\phi_{cm}$. The invariants s, t, and u are efined by

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p_2 + p_4)^2, && (7)\\
t &= (p_3 - p_1)^2 = (p_2 - p_4)^2, && (8)\\
u &= (p_4 - p_1)^2 = (p_2 - p_3)^2, && (9)
\end{aligned}
$$

where p_i are the 4-momentums. Momentum and energy conservation implies:

$$p_1 + p_2 = p_3 + p_4 \tag{10}$$

which guarantees that

$$s + t + u = \sum_{i=1}^{4} m_i^2. \tag{11}$$

The invariant s is the square of the total energy in the cnter of mass frame and is also given by

$$s = m_1^2 + m_2^2 + 2m_2(E_1)_{LAB}. \tag{12}$$

The invariant t is the 4-momentum transfer squared and is given by

$$t = t_{min} + 2p_{cm}p'_{cm}(\cos\theta_{cm} - 1), \tag{13}$$

where

$$t_{min} = (E_1^{cm} - E_3^{cm})^2 - (p_{cm} - p'_{cm})^2, \tag{14}$$

where the initial and final center-of-mass momenta are given by

$$
\begin{aligned}
p_{cm}^2 &= \frac{1}{4s}[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2], && (15)\\
p_{cm}'^2 &= \frac{1}{4s}[s - (m_3 + m_4)^2][s - (m_3 - m_4)^2], && (16)
\end{aligned}
$$

respectively, and the energies of particle 1 and 3 in the center-of-mass frame are

$$
\begin{aligned}
E_1^{cm} &= \frac{(s + m_1^2 - m_2^2)}{2\sqrt{s}} && (17)\\
E_3^{cm} &= \frac{(s + m_3^2 - m_4^2)}{2\sqrt{s}} && (18)
\end{aligned}
$$

The flux factor in eq.(5) is given by

$$[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2} = m_2(p_1)_{LAB} \tag{19}$$

$$= p_{cm}\sqrt{s} \tag{20}$$

in the laboratory and center-of-mass frame, respectively. Using eq.(6), we have

$$\frac{d\sigma}{dt}(s,t) = \frac{\pi}{p_{cm}p'_{cm}}\frac{d\sigma}{d\Omega_{cm}}(s,\theta) = \frac{1}{64\pi s p_{cm}^2} \mid \mathcal{M}_{\beta\alpha} \mid^2 . \tag{21}$$

The Kibble function is given by

$$\begin{aligned}\phi(s,t,u) &= stu - s(m_1^2 m_2^2 + m_3^2 m_4^2) \\ &- t(m_1^2 m_3^2 + m_2^2 m_4^2) - u(m_1^2 m_4^2 + m_2^2 m_3^2) \\ &+ 2(m_1^2 m_2^2 m_3^2 + m_1^2 m_3^2 m_4^2 + m_2^2 m_3^2 m_4^2 + m_1^2 m_2^2 m_4^2).\end{aligned} \tag{22}$$

The solution of implicit equation $\phi(s,t,u) = 0$ gives the boundaries of the physical phase space region for the reaction shown in Fig.

2 Metric and Matrices

B.1 Metric ($N = 4$ dimensions).

The metric

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{23}$$

or

$$g_{00} = 1, \quad g_{0i} = 0, \quad g_{ij} = -\delta_{ij} \quad (i,j = 1,3) \tag{24}$$

is chosen throughout the text. The space-time and energy-momentum four vectors are taken to be

$$x_\mu = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{25}$$

$$p_\mu = \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} \tag{26}$$

and the 4-vector dot product is given by

$$p^2 \equiv p \cdot p = g_{\mu\nu} p_\mu p_\nu = E^2 - \vec{p} \cdot \vec{p} \tag{27}$$

and

$$p_1 p_2 \equiv g_{\mu\nu} (p_1)_\mu (p_2)_\nu = E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2. \tag{28}$$

Repeated indices are sumed.

B.2 Metric (N dimensions).

In N spacetime (one time, $N-1$ space) dimensions the metric is taken to be

$$g_{00} = 1, \qquad g_{0i} = 0, \qquad g_{ij} = -\delta_{ij} \qquad (i, j = 1, 2, .., N-1) \tag{29}$$

and the N- vector dot product is

$$k \cdot p \equiv g_{\mu\nu} k_\mu p_\nu = k_0 p_o - k_1 p_1 - k_2 p_2 - .. - k_{N-1} p_{N-1}. \tag{30}$$

B.3 Dirac Algebra ($N = 4$ dimensions).

The 4×4 Dirac matrices satisfy

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = \gamma_\mu, \gamma_\nu = 2 g_{\mu\nu} I_4 \tag{31}$$

where I_4 is the 4×4 unit matrix and μ and ν run over $0, 1, 2, 3$.

$$\gamma_\mu \gamma_\mu = 4I_4 \tag{32}$$

$$\gamma_\mu \gamma_\alpha \gamma_\mu = -2\gamma_\alpha \tag{33}$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\mu = 4g_{\alpha\beta} I_4 \tag{34}$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\lambda \gamma_\mu = -2\gamma_\lambda \gamma_\beta \gamma_\alpha \tag{35}$$

Furthermore, the following trace relations are satisfied

$$tr(I_4) = 4, \tag{36}$$

$$tr(\gamma_\alpha \gamma_\beta) = 4g_{\alpha\beta} \tag{37}$$

$$tr(odd\ of\ \gamma' s) = 0 \tag{38}$$

$$tr(\gamma_\alpha \gamma_\beta \gamma_\lambda \gamma_\rho) = 4(g_{\alpha\beta} g_{\lambda\rho} - g_{\alpha\lambda} g_{\beta\rho} + g_{\alpha\rho} g_{\beta\lambda}) \tag{39}$$

In deriving the above relations one makes use of

$$tr(AB) = tr(BA), \tag{40}$$

$$tr(ABC) = tr(CAB) = tr(BCA), \tag{41}$$

$$tr(c_1 A + c_2 B) = c_1 tr(A) + c_2 tr(B) \tag{42}$$

where A , B, C are matrices and c_1, c_2 are constants.

The symbol $\hat{a}$ is defined by

$$\hat{a} \equiv a \cdot \gamma = g_{\mu\nu} a_\mu \gamma_\nu = a_0 \gamma_0 - \vec{a} \cdot \vec{\gamma}, \tag{43}$$

where a is an arbritrary 4-vector. ¿From above equations

$$tr(\hat{a}\hat{b}) = 4a \cdot b \tag{44}$$

$$tr(\hat{a}\hat{b}\hat{c}\hat{d}) = 4[a \cdot bc \cdot d - a \cdot cb \cdot d + a \cdot db \cdot c] \tag{45}$$

B.4 Dirac Matrices ($N = 4$ dimensions).

A convenient representation of the Dirac algebra in eq.(31) is given by

$$\gamma_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \tag{46}$$

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad (i = 1, 3), \tag{47}$$

where I_2 is the 2×2 unit matrix and the Pauli matrices, σ_i, satisfy

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{48}$$

$$\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k \tag{49}$$

and are represented by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{50}$$

It is usefull to define

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \tag{51}$$

and

$$\sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) = \frac{i}{2}[\gamma_\mu\gamma_\nu], \tag{52}$$

where

$$\gamma_5^2 = I_4 \tag{53}$$

and

$$\{\gamma_\mu, \gamma_5\} = 0 \tag{54}$$

It is easy to see that

$$tr(\gamma_\mu) = 0 \tag{55}$$

B.5 Dirac Algebra (N dimensions).

The N (one time, $N-1$ space) Dirac matrices satisfy

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}I_N \tag{56}$$

where I_N is the unit natrix in N dimensions and μ and ν run from 0 to $N-1$.

In addition

$$\begin{aligned}
\gamma_\mu \gamma_\mu &= N I_N && (57)\\
\gamma_\mu \gamma_\alpha \gamma_\mu &= (2-N)\gamma_\alpha && (58)\\
\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\mu &= 4 g_{\alpha\beta} I_N + (N-4)\gamma_\alpha \gamma_\beta && (59)\\
\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\lambda \gamma_\mu &= -2\gamma_\lambda \gamma_\beta \gamma_\alpha - (N-4)\gamma_\alpha \gamma_\beta \gamma_\lambda && (60)
\end{aligned}$$

Furthermore, the trace of I_N is defined as follows:

the following trace relations are satisfied

$$tr(I_N) = 4, \tag{61}$$

resulting in

$$\begin{aligned}
tr(\gamma_\alpha \gamma_\beta) &= 4 g_{\alpha\beta} && (62)\\
tr(odd\ of\ \gamma' s) &= 0 && (63)\\
tr(\gamma_\alpha \gamma_\beta \gamma_\lambda \gamma_\rho) &= 4(g_{\alpha\beta} g_{\lambda\rho} - g_{\alpha\lambda} g_{\beta\rho} + g_{\alpha\rho} g_{\beta\lambda} && (64)
\end{aligned}$$

The symbol $\hat{a}$ is defined by

$$\hat{a} \equiv a \cdot \gamma = g_{\mu\nu} a_\mu \gamma_\nu. \tag{65}$$

¿From above equations

$$\begin{aligned}
tr(\hat{a}\hat{b}) &= 4 a \cdot b && (66)\\
tr(\hat{a}\hat{b}\hat{c}\hat{d}) &= 4[a \cdot bc \cdot d - a \cdot cb \cdot d + a \cdot db \cdot c] && (67)
\end{aligned}$$

B.6 Dirac Equation and Spinors.

A free spin - 1/2 particle of mass m and spin s is described by the spinor $u(p,s)$ which satisfies the Dirac equation

$$(\hat{p} - m) u(p,s) = 0, \tag{68}$$

while the adjoint spinor

$$\bar{u}(p,s) \equiv u^{\dagger}(p,s)\gamma_0 \tag{69}$$

satisfies

$$\bar{u}(p,s)(\hat{p}-m) = 0 \tag{70}$$

$$u_{\lambda}(p) = \sqrt{E+m}\begin{pmatrix} \chi \\ \frac{\vec{\sigma}\vec{p}}{E+m}\chi \end{pmatrix} \tag{71}$$

The spin 4-vector s satisfies

$$s \cdot p \quad = \quad 0 \tag{72}$$

$$s^2 \quad = -1 \tag{73}$$

and in the rest frame of particle has the form

$$s_{\mu} = \begin{pmatrix} 0 \\ \vec{s} \end{pmatrix} \tag{74}$$

where $\vec{s}$ is the polarization vector and

$$\vec{s} \cdot \vec{s} = 1$$

.

The antiparticle spinors satisfy

$$(\hat{p}+m)v(p,s) = 0, \tag{75}$$

$$\bar{v}(p,s)(\hat{p}+m) = 0, \tag{76}$$

while the adjoint spinor

$$\bar{v}(p,s) \equiv v^{\dagger}(p,s)\gamma_0 \tag{77}$$

satisfies

$$\bar{v}(p,s)(\hat{p}+m) = 0 \tag{78}$$

The spinors and antispinors are normalized so that

$$\sum_{spin} \bar{u}(p,s)u(p,s) = 2m \tag{79}$$

$$\sum_{spin} \bar{v}(p,s)v(p,s) = -2m \tag{80}$$

$$\tag{81}$$

and the projection operators $\Lambda_\pm$ are given by

$$2m\Lambda_+ = \sum_{spin} u(p,s)\bar{u}(p,s) = \hat{p} + m \tag{82}$$

$$2m\Lambda_- = -\sum_{spin} v(p,s)\bar{v}(p,s) = -\hat{p} + m \tag{83}$$

$$\tag{84}$$

In forming the absolute value squared of the scattering amplitude one encounters the Hermitian conjugate which can be expressed as

$$[\bar{u}(p',s')\Gamma u(p,s)]^\dagger = \bar{u}(p,s)\bar{\Gamma}u(p',s') \tag{85}$$

where

$$\bar{\Gamma} = \gamma_0 \Gamma^\dagger \gamma_0 \tag{86}$$

Some examples are

$$\bar{\gamma}_\mu = \gamma_\mu \tag{87}$$

$$\bar{\sigma}_{\mu\nu} = \sigma_{\mu\nu} \tag{88}$$

$$\overline{(i\gamma_5)} = (i\gamma_5) \tag{89}$$

$$\overline{\hat{p}_1\hat{p}_2...\hat{p}_n} = \hat{p}_n\hat{p}_{n-1}...\hat{p}_1 \tag{90}$$

A convenient basis for the spin is the helicity basis in which $u_\lambda(p)$ are eigenstates of the operator $\frac{1}{2}\gamma_5 s$, with eigenvectors $\lambda = \pm\frac{1}{2}$ corresponding to spin aligned parallel or antiparallel to the direction of motion. In this case eq.(81) imply

$$\bar{u}_\lambda(p)u_{\lambda'}(p) = 2m\delta_{\lambda\lambda'} \tag{91}$$

$$\bar{v}_\lambda(p)v_{\lambda'}(p) = -2m\delta_{\lambda\lambda'} \tag{92}$$

An explicit form for the spinor representation of a fermion with momentum $\vec{p}$ along the positive $\hat{z}$ - axis and with helicity λ is given by:

$$u_\lambda(p) = \sqrt{E+m}\begin{pmatrix} \chi_\lambda \\ \frac{2\lambda|\vec{p}|}{E+m}\chi_\lambda \end{pmatrix} \tag{93}$$

$$\bar{u}_\lambda(p) = \sqrt{E+m}\left(\chi_\lambda^\dagger, \quad \frac{-2\lambda|\vec{p}|}{E+m}\chi_\lambda^\dagger\right) \tag{94}$$

where

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{95}$$

and $E^2 = p^2 + m^2$. The corresponding antiparticle is computed using

$$v_\lambda(p) = (-1)^{\lambda-\frac{1}{2}}\gamma_5 u_\lambda(p) \tag{96}$$

3 Feynman Rules and Integrals

C.1 Feynman Rules - General Discussion

External lines:

For particles in the initial or final state one writes the following factor:

(a) Spin Zero Boson 1

(b) Spin One Boson . . . . $\epsilon_\mu(\lambda)$.
Here $\epsilon_\mu(\lambda)$ is the polarization 4-vector for a boson with helicity λ. For the case of a massless spin one boson propagating along the z - axis with 4-momentum k_μ given by:

$$k_\mu = \begin{pmatrix} k_0 \\ 0 \\ 0 \\ k_3 \end{pmatrix} \tag{97}$$

with $k_0 = |k_3|$, the polarization 4-vectors are given by

$$\epsilon_\mu(\lambda = \pm 1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \tag{98}$$

for helicity ± 1. The polarization 4 - vector satisfy

$$k \cdot \epsilon = 0 \tag{99}$$

and

$$\epsilon^2 = -1 \tag{100}$$

For a massive spin 1 boson with 4-momentum given by eq.(97) but with $k_0^2 = k_3^2 + M^2$ one also has the longitudinal state

$$\epsilon_\mu(\lambda = 0) = \frac{1}{M} \begin{pmatrix} k_3 \\ 0 \\ 0 \\ k_0 \end{pmatrix} \tag{101}$$

where M is the boson mass.

(c) Spin 1/2 fermion of momentum p and spin s

in initial state . . . . . $u(p, s)$ on the right

in final state . . . . . $\bar{u}(p, s)$ on the left

(d) Spin 1/2 antifermion of momentum p and spin s

in initial state . . . . . $\bar{v}(p, s)$ on the left

in final state . . . . . $v(p, s)$ on the right

Internal Lines (Propagators):

Each internal line describes a particle of momentum q and mass m. Some examples are as follows:

(a) Spin Zero Boson

$$\frac{i}{q^2 - m^2 + i\epsilon} \tag{102}$$

(b) Photon (Feynman Gauge)

$$\frac{-ig_{\mu\nu}}{q^2 + i\epsilon} \tag{103}$$

(a) Spin One Boson

$$\frac{-i(g_{\mu\nu} - q_\mu q_\nu/m^2)}{q^2 - m^2 + i\epsilon} \tag{104}$$

(a) Spin 1/2 fermion

$$\frac{i(\hat{q} + m)}{q^2 - m^2 + i\epsilon} \tag{105}$$

For antifermions one uses the same propagators, treating the antifermion as a fermion of opposite 4-momentum (i.e., $-q$).

Vertex Factors:

For each intersection of three (or more) lines at one point there is a vertex factor which depends on the structure of the interaction Lagrangian. Some examples are shown in Fig.C.1.

(a) Three scalar boson vertex

$$-ig \tag{106}$$

(b) $ee\gamma$ vertex

$$-ie\gamma_\mu \tag{107}$$

Here e is the charge of the electron and the fine structure constant $\alpha = e^2/(4\pi)$.

(c) Charget spin zero boson-photon vertex

$$-iQ(p_1 + p_2)_\mu \tag{108}$$

(d) Four point coupling for charged spin zero boson-photon vertex

$$2iQ^2 g_{\mu\nu} \tag{109}$$

Loops and Combinatorics :

(a) For each loop with undetermined momentum k . . . . . $\int d^4k/(2\pi)^4$

Here the integral runs over all values of the momentum.

(b) For each closed fermion loop . . . . -1.

(b) For each closed loop containing n identical bosons . . . . 1/n!.

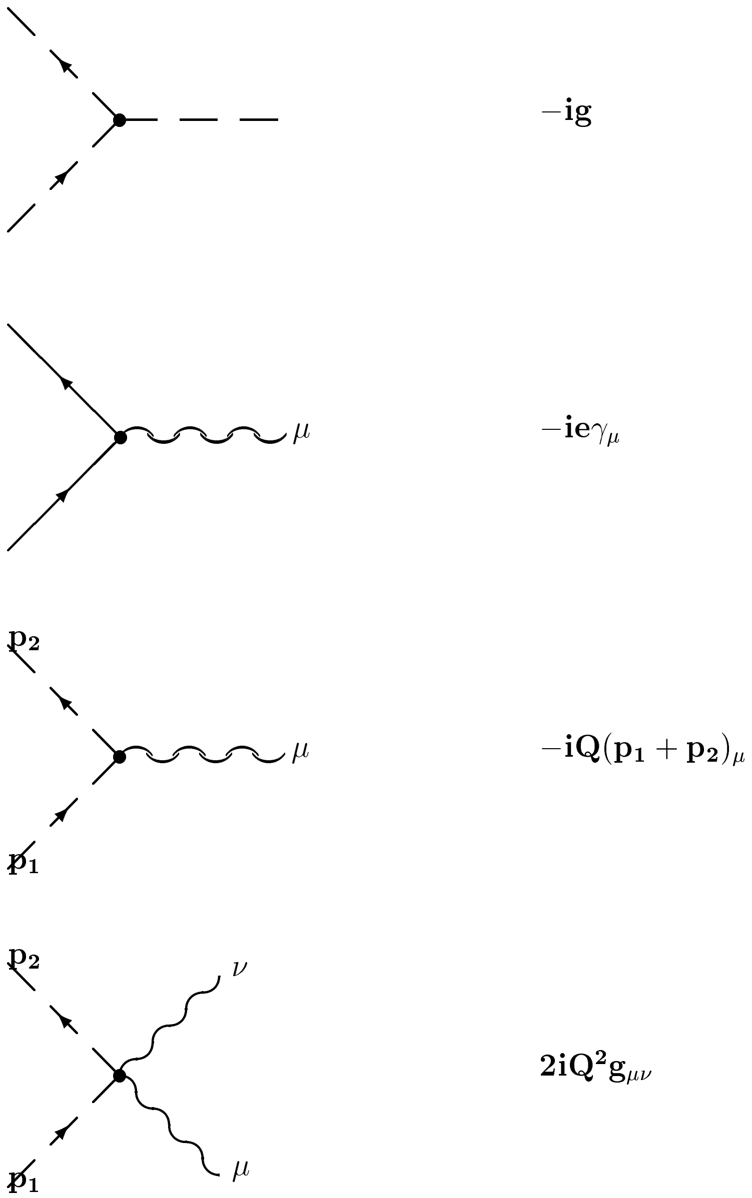


Fig. C.1 Couplings for the triple-scalar-boson, electron-photon, charged spin zero boson-photon, and the four point charged spin zero boson - photon - photon vertices.

C.2 Loop Integrations

Feynman Parameterizations:

$$\frac{1}{a_1 a_2 ... a_n} = \Gamma(n) \int_0^1 dx_1 \int_0^1 dx_2 ... \int_0^1 dx_n \frac{\delta(1 - \sum_{i=1}^n x_n)}{[\sum a_i x_i]^n} \tag{110}$$

$$\frac{1}{abc} = 2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\delta(1 - x - y - z)}{[ax + by + cz]^3} \tag{111}$$

$$\frac{1}{a^R b^M} = \frac{\Gamma(R+M)}{\Gamma(R)\Gamma(M)} \int_0^1 dx \int_0^1 dy \frac{x^{R-1} y^{M-1} \delta(1 - x - y)}{[ax + by]^{R+M}} \tag{112}$$

$$= \frac{\Gamma(R+M)}{\Gamma(R)\Gamma(M)} \int_0^1 dx \frac{x^{R-1}(1-x)^{M-1}}{[ax + b(1-x)]^{R+M}} \tag{113}$$

Integral over d^4K ($N = 4$ dimensions):

$$\int \frac{d^4 k}{(2\pi)^4} \frac{(K^2)^R}{(K^2 - C)^M} = \frac{i(-1)^{R-M}}{16\pi^2} C^{R-M+2} \frac{\Gamma(R+2)\Gamma(M-R-2)}{\Gamma(M)} \tag{114}$$

Integral over $d^N K$ (N dimensions):

$$\int \frac{d^N k}{(2\pi)^N} \frac{(K^2)^R}{(K^2 - C)^M} = \frac{i(-1)^{R-M}}{(16\pi^2)^{N/4}} C^{R-M+N/2} \frac{\Gamma(R+\frac{N}{2})\Gamma(M-R-\frac{N}{2})}{\Gamma(\frac{N}{2})\Gamma(M)} \tag{115}$$

C.3 Feynman Rules - QCD

External Lines

(a) Gluons . . . . $\epsilon_\mu(\lambda)$

Here $\epsilon_\mu(\lambda)$ is the polarization 4-vector for a gluon with helicity λ as in eq.(98). It satisfies the orthogonality condition

$$\epsilon_\mu(\lambda)\epsilon_\mu(\lambda') = -\delta_{\lambda\lambda'} \tag{116}$$

and

$$\epsilon \cdot k = 0 \tag{117}$$

where k_μ is the 4-momentum of the gluon.

(b) Quark of momentum p and spin s

in initial state . . . . . $u(p,s)$ on the right

in final state . . . . . $\bar{u}(p,s)$ on the left

(d) Antiquark of momentum p and spin s

in initial state . . . . . $\bar{v}(p,s)$ on the left

in final state . . . . . $v(p,s)$ on the right

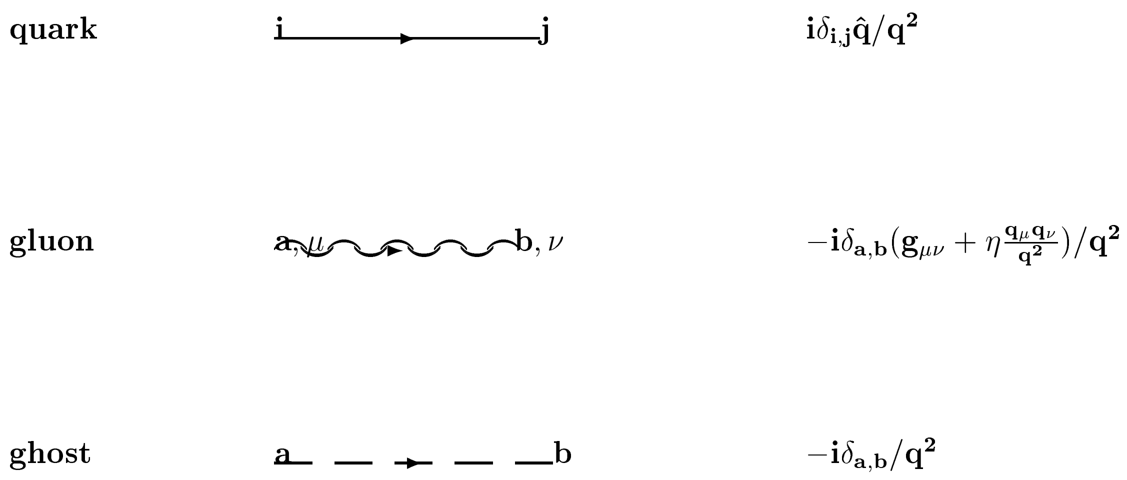


Fig. C.2 QCD propagators

Internal Lines (Propagators):

The QCD propagators shown in Fig. C.2 are the following:

(a) Quarks

$$\frac{i\delta_{ij}\hat{q}}{q^2 + i\epsilon} \tag{118}$$

Here I have the quark mass to be zero. The indices i and j correspond to quark color and run from 1 to 3.

(b) Gluons (covariant gauge)

$$\frac{-i\delta_{ab}[(g_{\mu\nu} - q_\mu q_\nu/q^2) + \beta q_\mu q_\nu/q^2]}{q^2 + i\epsilon} \tag{119}$$

or

$$\frac{-i\delta_{ab}(g_{\mu\nu} + \eta q_\mu q_\nu/q^2)}{q^2 + i\epsilon} \tag{120}$$

Here the $\eta \equiv \beta - 1$ is an arbitrary gauge parameter ($\eta = 0$ is the Feynman gauge, $\eta = -1$ is the Landau gauge). The indices a and b correspond to the gluon color and run from 1 to 8.

(c) Ghost

$$\frac{-i\delta_{ab}}{q^2} \tag{121}$$

In a covariant gauge one must include internal ghost particle that couple to gluons as shown in Fig.C.2.

Vertex Factors:

The QCD vertex factors shown in Fig.C.3 are as follows:

(a) Quark - gluon - gluon

$$-ig_s\gamma_\mu T^a_{ij} \tag{122}$$

The indices i and j correspond to the quark color and run from 1 to 3 and a corresponds to the gluon color and runs from 1 to 8. The eight 3×3 $SU(3)$ matrices, T_a, handle the color and will be discussed in Appendix D. The QCD coupling constant is given by:

$$\alpha = \frac{g_s^2}{4\pi} \tag{123}$$

(b) Triple-gluon coupling

$$V_{\lambda\mu\nu}(p_1, p_2, p_3) = -g_s f_{abc} F_{\lambda\mu\nu}(p_1, p_2, p_3) \tag{124}$$

where

$$F_{\lambda\mu\nu}(p_1, p_2, p_3) = (p_1 - p_2)_\nu g_{\lambda\mu} + (p_2 - p_3)_\lambda g_{\mu\nu} + (p_3 - p_1)_\mu g_{\nu\lambda} \tag{125}$$

The indices a, b, c correspond to the gluon color and run from 1 to 8 and f_{abc} are the $SU(3)$ structure constants discussed in Appendix D. Energy momentum conservation inplies $p_1 + p_2 + p_3 = 0$.

(c) Ghost - ghost - gluon coupling

$$g_s f_{abc} p_\mu \tag{126}$$

The indices a, b, c correspond to color and run from 1 to 8 and f_{abc} are the $SU(3)$ structure constants discussed in Appendix D.

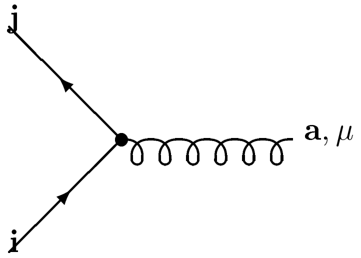
(d) Four - gluon coupling

$$\begin{aligned}
&-ig_s^2 f_{abc} f_{cde}(g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\sigma} g_{\mu\nu}) \\
&-ig_s^2 f_{ace} f_{bde}(g_{\lambda\mu} g_{\nu\sigma} - g_{\lambda\sigma} g_{\mu\nu}) \\
&-ig_s^2 f_{ade} f_{cbe}(g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\mu} g_{\sigma\nu})
\end{aligned} \tag{127}$$

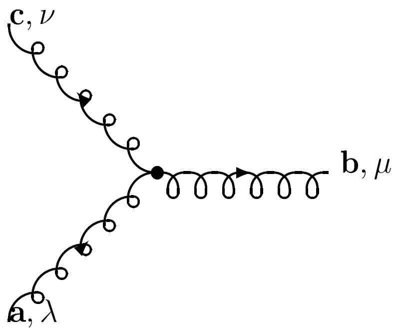
The indices a, b, c d correspond to color and run from 1 to 8 and f_{abc} are the $SU(3)$ structure constants discussed in Appendix D.

Loops and Combinatorics:

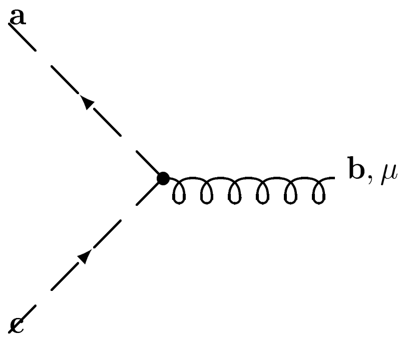
Same as given in Appendix C.1 except that for ghost loops one must include a factor of -1.



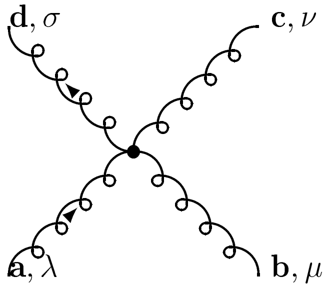
$$-\mathbf{i}\mathbf{g}_\mathbf{s}\gamma_\mu \mathbf{T}^\mathbf{a}_\mathbf{i,j}$$



$$-\mathbf{i}\mathbf{g}_\mathbf{s}\mathbf{f}_\mathbf{abc}[(\mathbf{p_1}-\mathbf{p_2})_\nu \mathbf{g}_{\lambda\mu} + (\mathbf{p_2}-\mathbf{p_3})_\lambda \mathbf{g}_{\mu\nu} + (\mathbf{p_3}-\mathbf{p_1})_\mu \mathbf{g}_{\nu\lambda}]$$



$$\mathbf{g}_\mathbf{s}\mathbf{f}_\mathbf{abc}\mathbf{p}_\mu$$



$$-\mathbf{i}\mathbf{g}_\mathbf{s}^\mathbf{2}\mathbf{f}_\mathbf{abc}\mathbf{f}_\mathbf{cde}(\mathbf{g}_{\lambda\nu}\mathbf{g}_{\mu\sigma} - \mathbf{g}_{\lambda\sigma}\mathbf{g}_{\mu\nu})$$
$$-\mathbf{i}\mathbf{g}_\mathbf{s}^\mathbf{2}\mathbf{f}_\mathbf{ace}\mathbf{f}_\mathbf{bde}(\mathbf{g}_{\lambda\mu}\mathbf{g}_{\nu\sigma} - \mathbf{g}_{\lambda\sigma}\mathbf{g}_{\mu\nu})$$
$$-\mathbf{i}\mathbf{g}_\mathbf{s}^\mathbf{2}\mathbf{f}_\mathbf{ade}\mathbf{f}_\mathbf{cbe}(\mathbf{g}_{\lambda\nu}\mathbf{g}_{\mu\sigma} - \mathbf{g}_{\lambda\mu}\mathbf{g}_{\sigma\nu})$$

Fig. C.3 QCD vertex factors. Quark-quark-gluon, triple-gluon, ghost-ghost-gluon, four - gluon.

4 SU(3) of Color

4.1 D.1 Structure Constants and Color Matrices

The 3x3 $SU(3)$ color matrices , T_a, satisfy

$$[T_a, T_b] = i f_{abc} T_c \tag{128}$$

where f_{abc} are the antisymmetric $SU(3)$ structure constants with non-zero values given by:

$$\begin{array}{cccc} a & b & c & f_{abc} \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 7 & \frac{1}{2} \\ 1 & 5 & 6 & \frac{-1}{2} \\ 2 & 4 & 6 & \frac{1}{2} \\ 2 & 5 & 7 & \frac{1}{2} \\ 3 & 4 & 5 & \frac{1}{2} \\ 3 & 6 & 7 & \frac{-1}{2} \\ 4 & 5 & 8 & \frac{\sqrt{3}}{2} \\ 6 & 7 & 8 & \frac{\sqrt{3}}{2} \end{array} \tag{129}$$

The convenient representation of the T_a matrices is the one introduced by Gell-Mann in which:

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{130}$$

$$T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_4 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$T_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T_6 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$T_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_8 = \tfrac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

The structure constants d_{abc} are defined according to

$$T_a, T_b = \frac{1}{3}\delta_{ab} + d_{abc}T_c \tag{131}$$

The T_a matrices satisfy

$$T_aT_b = \frac{1}{2}[\frac{1}{3}\delta_{ab} + (d_{abc} + if_{abc})T_c] \tag{132}$$

$$T^a_{ij}T^a_{kl} = \frac{1}{2}[\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl}] \tag{133}$$

$$Tr(T_a) = 0, \tag{134}$$

$$Tr(T_aT_b) = \frac{1}{2}\delta_{ab}, \tag{135}$$

$$Tr(T_aT_bT_c) = \frac{1}{4}(d_{abc} + if_{abc}), \tag{136}$$

$$Tr(T_aT_bT_aT_c) = \frac{-1}{12}\delta_{bc}, \tag{137}$$

The structure constants satisfy the following Jacobi identities:

$$f_{abe}f_{ecd} + f_{cbe}f_{aed} + f_{dbe}f_{ace} = 0 \tag{138}$$

$$f_{abe}d_{ecd} + f_{cbe}d_{ced} + f_{dbe}d_{ace} = 0 \tag{139}$$

In addition,

$$f_{abc}f_{cde} = \frac{2}{3}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + (d_{ace}d_{bde} - d_{bce}d_{ade}) \tag{140}$$

It is sometime useful to define $8x8$ matrices F_a and D_a such that

$$(F_a)_{bc} = -if_{abc} \tag{141}$$

$$(D_a)_{bc} = d_{abc} \tag{142}$$

in which case the Jacobi identities (D.1.11) and (D.1.12) become

$$[F_a, F_b] = if_{abc}F_c \tag{143}$$

$$[F_a, D_b] = if_{abc}D_c \tag{144}$$

In addition $f_{abb} = 0$ and $d_{abb} = 0$ implies:

$$tr(F_a) = 0 \tag{145}$$

$$tr(D_a) = 0 \tag{146}$$

The following are some useful relationships

$$f_{acd}f_{bcd} = 3\delta_{ab} \tag{147}$$

$$F_aF_a = 3 \tag{148}$$

$$tr(F_aF_b) = 3\delta_{ab} \tag{149}$$

$$f_{acd}d_{bcd} = 0 \tag{150}$$

$$F_aD_a = 0 \tag{151}$$

$$tr(F_aD_b) = 0 \tag{152}$$

$$d_{acb}d_{bcd} = \frac{5}{3}\delta_{ab} \tag{153}$$

$$D_a D_a = \frac{5}{3} \tag{154}$$

$$tr(D_a D_b) = \frac{5}{3}\delta_{ab} \tag{155}$$

$$tr(F_a F_b F_c) = i\frac{3}{2}f_{abc} \tag{156}$$

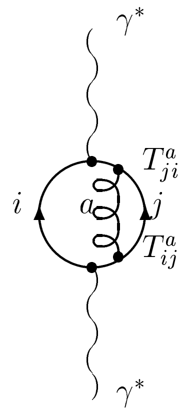
$$tr(D_a F_b F_c) = \frac{3}{2}d_{abc} \tag{157}$$

$$tr(D_a D_b F_c) = i\frac{5}{6}f_{abc} \tag{158}$$

$$tr(D_a D_b D_c) = -\frac{1}{2}d_{abc} \tag{159}$$

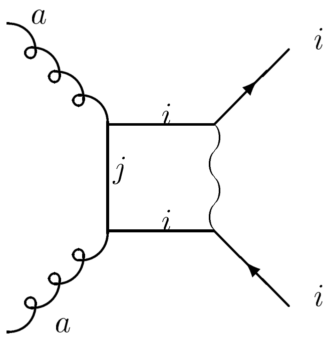
$$tr(F_a F_b F_a F_c) = \frac{9}{2}\delta_{bc} \tag{160}$$

(a)



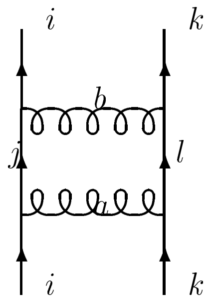
$T_r(T_a T_a) = 4$

(b)



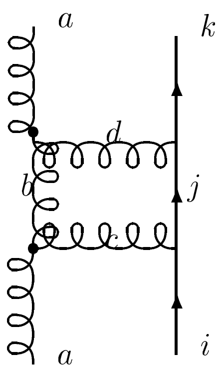
$\frac{1}{8\times 3} T_r(T_a T_a) = \frac{1}{6}$

(c)



$\frac{1}{3\times 3} T_r(T_a T_b) T_r(T_a T_b) = \frac{2}{9}$

(d)



$\frac{1}{3\times 8} f_{abc} f_{bad} T_r(T_c T_d) = \frac{1}{2}$

Fig D.1 Color factors for the square of the amplitudes (a) $\gamma^* \to q+\bar{q}$, (b) $g+q \to q+\gamma$, (c) $q+q \to q+q$, and (d) $g+q \to g+q$.

4.2 Examples of Color Factors

As is the case with particle spins, we choose the sum over the final color states and average over initial color states. The color factor for the square of the $\gamma^* \to q\bar{q}g$ amplitude shown in Fig.D.1a is given by:

$$T^a_{ij}T^a_{ji} = tr(T_aT_a) = \frac{1}{2}\delta_{aa} = 4, \tag{161}$$

since

$$\delta_{aa} = 8. \tag{162}$$

Similarly, the color factor for the square of the $gq \to \gamma^* q$ amplitude shown in Fig.D.1b is given by:

$$\frac{1}{8}\frac{1}{3}T^a_{ij}T^a_{ji} = \frac{1}{24}tr(T_aT_a) = \frac{1}{6}, \tag{163}$$

where the factor of $\frac{1}{24}$ arises from the initial state color averaging.

The color factor for square of the $qq \to qq$ amplitude shown in Fig.D.1c is given:

$$\begin{aligned} \frac{1}{3}\frac{1}{3}T^a_{ij}T^b_{ji}T^a_{kl}T^b_{lk} &= \frac{1}{9}tr(T_aT_b)tr(T_aT_b) \\ &= \frac{1}{9}(\frac{1}{2}\delta_{ab})(\frac{1}{2}\delta_{ab}) = \frac{1}{9}(\frac{1}{4}\delta_{aa} = \frac{2}{9}. \end{aligned} \tag{164}$$

The color factor for the square of the $gq \to gq$ amplitude shown in Fig.D1.d is given by:

$$\begin{aligned} \frac{1}{8}\frac{1}{3}f_{abc}f_{bad}T^c_{ij}T^d_{ji} &= \frac{1}{24}f_{abc}f_{bad}tr(T_cT_d) \\ &= \frac{1}{24}(\frac{1}{2}\delta_{cd})f_{abc}f_{bad} = \frac{1}{48}f_{abc}f_{bad} \\ &= \frac{1}{48}tr(F_aF_a) = \frac{1}{48}3\delta_{aa} = \frac{1}{2}, \end{aligned} \tag{165}$$

where eq.(135 and (130) have been used.

5 Special Functions

5.1 Gamma Function

The gamma function $\Gamma(x)$ is defined by:

$$\Gamma(x) = \int_0^1 e^{-y} y^{x-1} dy, \quad (x > 0) \tag{166}$$

and has the properties

$$\Gamma(n+1) = n! \quad (n = integer) \tag{167}$$

$$\Gamma(x+1) = x\Gamma(x) \quad (x > 0) \tag{168}$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{sin(\pi x)} \quad (x > 0) \tag{169}$$

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(x+\frac{1}{2}) \tag{170}$$

$$\Gamma(1) = 1 \tag{171}$$

$$\Gamma(1/2) = \sqrt{\pi} \tag{172}$$

$$\lim_{x\to\infty} \Gamma(x+y+1) = \Gamma(x+1)x^y \tag{173}$$

$$\lim_{x\to\infty} \Gamma(x+1) = \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} \tag{174}$$

$$\lim_{x\to\infty} \left(\frac{\Gamma(x+y)}{\Gamma(x)} \right) = x^y \tag{175}$$

$$\lim_{x\to\infty} \left(\frac{\Gamma(x+y)}{\Gamma(x+b)} \right) = x^{a-b} \tag{176}$$

In addition,

$$B(R,M) = \int_0^1 dx x^{R-1}(1-x)^{M-1} = \frac{\Gamma(R)\Gamma(M)}{\Gamma(R+M)} \tag{177}$$

is the beta function and

$$\int_0^1 dx x^{C-1}(1-x)^{M-1} \int_0^1 dy y^{B-1}(1-y)^{C-B-1}(1-xy)^{-A} =$$

$$= \frac{\Gamma(B)\Gamma(M)\Gamma(C-B)\Gamma(C+M-A-B)}{\Gamma(C+M-A)\Gamma(C+M-B)} \tag{178}$$

where the latter is valid provided $Re(C) > 0$, $Re(B) > 0$ and $Re(C+M-A-B) > 0$.

THe first and second derivatives of the gamma function evaluated at $x = 1$ are given by:

$$\Gamma'(1) = -\gamma_E \tag{179}$$

$$\Gamma''(1) = \gamma_E^2 + \frac{1}{6}\pi^2 \tag{180}$$

respectively, where γ_E is Eulers constant which satisfies:

$$\gamma_E = \lim_{n\to\infty} \left[-log(n) + 1 + \frac{1}{2} + \frac{1}{3} + .. + \frac{1}{n}\right] = 0.5772157 \tag{181}$$

and

$$\gamma_E = -\int_0^\infty e^{-x} log(x) dx \tag{182}$$

¿From eq.(179) and (180) we have the following Taylor series expansion of $\Gamma(1+\epsilon)$,

$$\Gamma(1+\epsilon) = 1 - \gamma_E\epsilon + \frac{1}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right)\epsilon^2 + .. \tag{183}$$

The Taylor expansion of the logarithm of the gamma function is

$$log[\Gamma(1+\epsilon)] = -\gamma_E\epsilon + \frac{1}{2}\rho(2)\epsilon^2 - \frac{1}{3}\rho(3)\epsilon^3 + ... \tag{184}$$

where

$$\rho(p) = 1 + \frac{1}{2^p} + \frac{1}{3^p} + ... \tag{185}$$

is the zeta function of order p with

$$\rho(2) = \frac{1}{6}\pi^2 \tag{186}$$

5.2 Dilogarithm

A function that appears in many QCD perturbative calculations is the dilogarithm, $Li_2(x)$. It is defined by

$$Li_2(x) = \sum_{n=1}^{\infty} \frac{x^2}{n^2}, \quad (|x| < 1) \tag{187}$$

or

$$Li_2(x) = -\int_0^x \frac{log(1-t)}{t} dt, \tag{188}$$

Equivalent definitions are given by

$$Li_2(x) = -\int_0^1 \frac{log(1-xt)}{t} dt, \tag{189}$$

$$Li_2(x) = -\int_{1-x}^1 \frac{log(t)}{1-t} dt, \tag{190}$$

$$Li_2(x) = \int_0^1 \frac{log(t)}{t-\frac{1}{x}} dt, \tag{191}$$

The dilogarithm has the following properties

$$Li_2(0) = 0 \tag{192}$$

$$Li_2(1) = \rho(2) = \frac{\pi^2}{6} \tag{193}$$

$$Li_2(-1) = -\frac{\pi^2}{12} \tag{194}$$

$$Li_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{1}{2} log^2(2) \tag{195}$$

$$Li_2(-x) + Li_2(-\frac{1}{x}) = -\frac{\pi^2}{6} - \frac{1}{2} log^2(x) \quad (x > 0) \tag{196}$$

$$Li_2(x) + Li_2(\frac{1}{x}) = \frac{\pi^2}{3} - \frac{1}{2} log^2(x) - i\pi log(x) \quad (x > 1) \tag{197}$$

$$Li_2(x) + Li_2(1-x) = \frac{\pi^2}{6} - log(x) log(1-x) \tag{198}$$

5.3 "+ Functions"

Strictly speaking "+ functions" are distributions that are well behaved only when convoulted with a smooth function that vanishes sufficiently rapidly as $x \to 1$. They have a property that

$$\int_0^1 (F(x))_+ dx = 0 \tag{199}$$

and are defined by trhe following limiting procedure

$$(F(x))_+ \equiv \lim_{\beta\to 0} \left\{ F(x)\theta(1-x-\beta) - \delta(1-x-\beta) \int_0^{1-\beta} F(y)dy \right\} \tag{200}$$

where

$$\theta(y) = 0 \quad fory \leq 0 \tag{201}$$

$$\theta(y) = 1 \quad fory > 0 \tag{202}$$

Two important "+ functions" are

$$\frac{1}{(1-x)_+} \equiv \lim_{\beta\to 0} \left\{ \frac{1}{1-x}\theta(1-x-\beta) + log(\beta)\delta(1-x-\beta) \right\} \tag{203}$$

and

$$\left(\frac{log(1-x)}{1-x}\right)_+ \equiv \lim_{\beta\to 0} \left\{ \frac{log(1-x)}{1-x}\theta(1-x-\beta) + \frac{1}{2}log^2(\beta)\delta(1-x-\beta) \right\}. \tag{204}$$

When convoluted with a well behaved function $G(y)$ one arrives at

$$\int_z^1 \frac{dy}{y}\frac{G(z/y)}{(1-y)_+} = G(z)log(1-z) + \int_z^1 \frac{dy}{y}\frac{[G(z/y) - yG(z)]}{(1-y)} \tag{205}$$

and

$$\int_z^1 \frac{dy}{y}G(z/y)\left(\frac{log(1-y)}{(1-y)}\right)_+ = \frac{1}{2}G(z)log^2(1-z) + \\ \int_z^1 \frac{dy}{y}\frac{[G(z/y) - yG(z)]}{(1-y)}log(1-y) \tag{206}$$

5.4 Gegenbauer Polinomials

The Gegenbauer polynomials, $C_n^{(\frac{3}{2})}(z)$, are orthogonal on the interval -1 to 1 with respect to the weight $(1-z^2)$. In particular

$$\int_{-1}^{1} dz(1-z^2)C_n^{(\frac{3}{2})}(z)C_m^{(\frac{3}{2})}(z) = \delta_{nm}\frac{(n+2)(n+1)}{2(2n+3)} \tag{207}$$

This can be expressed on the interval 0 to 1 by the change of variable $z = 2x-1$ and eq.(207) becomes

$$\int_{0}^{1} dxx(1-x)C_n^{(\frac{3}{2})}(2x-1)C_m^{(\frac{3}{2})}(2x-1) = \delta_{nm}\frac{(n+2)(n+1)}{4(2n+3)} \tag{208}$$

The Gegenbauer polinomials cab be deduced from the following recursive relation:

$$(n+1)C_{n+1}^{(\frac{3}{2})}(z) = (2n+3)zC_n^{(\frac{3}{2})}(z) - (n-2)C_{n-1}^{(\frac{3}{2})}(z) \tag{209}$$

where

$$C_0^{(\frac{3}{2})}(z) = 1 \tag{210}$$

$$C_1^{(\frac{3}{2})}(z) = 3z \tag{211}$$

It is useful to note that $1/(1-x)$ can be expanded in terms of Gegenbauer polynomials as follows:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}\frac{2(2n+3)}{(n+1)(n+3)}C_n^{(\frac{3}{2})}(2x-1) \tag{212}$$