Fall'01

Modern Physics

Name ____________________________

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- A-1. ✓ OM
- A-2. ✓ OM
- A-3. XW
- A-4. WH
- A-5. ✓ OM
- A-6. ✓ OM
- B-1. ✓ OM
- B-2. XW
- B-3. XZ

A-1. Three spaceships are equipped with lasers that produce monochromatic light of wavelength $\lambda_0 = 600\,\text{nm}$ in the laser rest frame. Spaceship A and spaceship C are moving away from spaceship B with equal speeds $v = 0.75c$ in *opposite* directions. The laser on ship A is turned on and directed towards ships B and C.

a) Determine the wavelength of the laser as seen by an observer on ship B.

b) Determine the wavelength of the laser as seen by an observer on ship C.

A-2. Consider the production of neutral pions ($m_\pi c^2 = 135$ MeV) in the collision of two protons ($m_p c^2 = 938.3$ MeV):

$$p + p \rightarrow p + p + \pi^0$$

In the center-of-mass frame the two incoming protons collide with equal and opposite mementa; in the laboratory frame, one of the protons is initially at rest. Using relativistic kinematics, calculate the threshold energy for this reaction, i.e., the minimum proton kinetic energy in the laboratory frame required for the reaction to take place.

A-3. An electron with a kinetic energy of 11.21 eV collides head-on with a ground-state hydrogen atom moving in the opposite direction. Before the collision, the total momentum of the system is zero.

a) What is the kinetic energy of the atom before collision?

b) As a result of the collision, the atom is excited into the n=2 state. What is the total kinetic energy of the system after the collision?

A-4. According to theoretical models, 4 billion years ago the planet Jupiter may have had a luminosity (total amount of energy radiated per unit time) that is as high as 10^{-2} that of the Sun and a surface temperature of 1000 K. The Sun's surface temperature is 5800 K and we may assume that this temperature has remained constant over the past 4 billion years. What was Jupiter's radius (as a fraction of the Sun's radius) at that time if both Jupiter and the Sun radiated as perfect blackbodies?

A-5. A sample of wood is burned in pure oxygen, yielding 200 g of CO_2 gas. The measured activity of the gas is R=180 decays per minute. (Note: M(C)=12 g/mol, M(O)=16 g/mol, and $N_{Avog} = 6.022\times10^{23}$).

a) Assuming that all the carbon originally in the wood ends up in the CO_2 gas, calculate the number of carbon nuclei in the sample.

b) ^{14}C has a half-life of 5730 years. Using the measured decay rate, determine the number of ^{14}C nuclei in the gas.

c) In the Earth's atmosphere, the $^{14}C/^{12}C$ equilibrium ration has the value of 1.3×10^{-12}. All living things on the Earth contain carbon atoms in this ratio, but after death, the ratio begins to decrease due to the radioactive decay of the ^{14}C nuclei (^{12}C is stable). Assuming that the wood sample was taken from a living tree or a tree shortly after its death, find the age of the sample.

A-6. Silver has mass density $\rho = 10.5\ \text{g/cm}^3$ and atomic weight M=107.9 g/mole.

a) Calculate the number of conduction electrons per cm^3 in silver assuming that each silver atom has one conduction electron. Note that Avogadro's number is 6.022×10^{23}.

b) The density of states in a free Fermi electron gas is given by the expression $g(E) = (4\pi V/h^3)(8m_e{}^3E)^{1/2}$, where E is the electron kinetic energy, h is Planck's constant ($hc = 1240\,\text{eV-nm}$), V is the volume occupied by the gas and m_e is the electron mass ($m_ec^2 = 0.511\,\text{MeV}$). Use this information and the result of part (a) to determine the Fermi energy (in eV) of the conduction electrons in silver at temperature T=0K.

c) Calculate the fraction of conduction electrons in silver at T=0K that have energies between the average electron energy and the Fermi energy.

B-1. Consider a particle in a one dimensional quantum mechanical harmonic oscillator potential. The wave function of a particle in the n=1 state of this oscillator has the form $\phi_1(x) = Ax\exp(-\alpha x^2)$ where A and α are constants.

a) Show that the time-independent Schrödinger's equation for a particle with wave function $\psi(x)$ in a one dimensional harmonic oscillator potential can be written in the form $d^2\psi/dx^2 = f(x)\psi(x)$ and determine the function $f(x)$.

b) Show that $\psi(x) = \phi_1(x)$ is a solution of this equation with an appropriate choice for α and determine the energy of the quantum mechanical state with this wave function.

c) Determine the constant A so that the wave function ϕ_1 is normalized.

(Note that $\int_{-\infty}^{\infty}\exp(-bx^2)dx = \sqrt{\pi/b}$)

B-2. Suppose that the potential acting on a proton ($m_p c^2 = 938.3\,\text{MeV}$) inside a nucleus is modeled as an infinite one-dimensional square well with boundaries at x=0 and x=L, where L=2.0 fm= 2.0×10^{-15} m.

a) Determine the energy of a proton (not including its mass energy) in the ground state of the well.
b) Calculate the wavelength of the photon that would be emitted if a proton makes an electromagnetic transition from the first excited state of the well to the ground state. Note that $hc = 1240\,\text{MeV-fm}$.
c) Determine the probability that a position measurement of a proton in the ground state of the well will located it within the region L/4<x<3L/4.

B-3. Consider an electron moving in a constant homogeneous magnetic field $\vec{B} = B\hat{z}$. There are no other potentials present.

a) Write down the Hamiltonian (Hint: canonical momentum is $\vec{p} + \frac{e}{c}\vec{A}$, where $\vec{A}$ is the vector potential.
b) Show that if one chooses the vector potential that only has y-component, the general solution of the wave function has plane-wave form in two of the three directions.
c) Find the allowed energy levels.

Fall 02

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A-1. X W
A-2. X W
A-3. X W
B-1. X
B-2. X
B-3. X
B-4. X
B-5. X
B-6. X

B

A-1. The commutation rules of the angular momentum operators (in units of $\hbar$) are $[J_k, J_l] = iJ_m$ with k,l,m being the cyclic permutations of x, y, z. Prove that $J^2 = J_x^2 + J_y^2 + J_z^2$ commute with each J_k and state what conclusions this leads to. What are the good quantum numbers for the Hamiltonian

$$H = \frac{p^2}{2m} + V(r) + \mu_0 B(J_z + S_z)$$

where B is a magnetic field, μ_0 is a constant and $\vec{J} = \vec{L} + \vec{S}$ with $\vec{S}$ being the spin operator? Shown details of the analysis leading to the answer.

A-2. A particle of mass m is constrained to move on a circle of radius R.

a) Find the energies and eigenfunctions of the lowest three eigenstates if the particle is subject to no other forces.
b) Suppose that the position of the particle on the circle is known within an accuracy $\Delta\theta$. Estimate the minimum energy that the particle must have.
c) Given that the wavefunction of the particle at $t = 0$ is $\psi(\theta) = 1 + \cos^2\theta$, find the wavefunction $\psi(\theta, t)$ at a later time t.

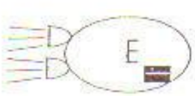
A-3. The normalized wavefunction for a quantum oscillator in its ground state is $\Psi(x) = \left(\frac{m\omega_0}{\hbar\pi}\right)^{1/4} \exp(-\frac{m\omega_0}{2\hbar}x^2)$. Determine the expectation values $\langle x \rangle$ and $\langle x^2 \rangle$, and from these the standard deviation, σ, in the position of the particle. (Hint: Ψ is an <u>even</u> function. You will probably need to know that the integral $\int_0^\infty x^2 \exp(-ax^2)dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$).

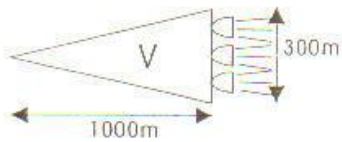
B-1. When a metal surface is illuminated with ultra-violet radiation of wavelength 283 nm, the stopping potential of the electrons that are emitted is 2.5 V.

a) What is the work function of the metal?
b) What is the threshold frequency of the metal?
c) If the wavelength is changed to 590 nm, what is the new stopping potential?

B-2. A Vogon spaceship is shaped like an isosceles triangle with proper dimensions as shown (i.e. when measured in its own rest frame). As it is approaches the Earth, an Earth ship sets out to intercept it. To observers on the Earth ship, the Vogon ship appears as an <u>equilateral</u> triangle of side 300 m.

a) What is the speed of the Vogon ship relative to the Earth ship?
b) If the Earth ship is moving at 0.8c relative to Earth, what is the speed of the Vogon ship relative to Earth?





B-3. For an ideal gas, the Maxwell-Boltzmann energy distribution of molecules is given by

$n(E)dE = \frac{2\pi N}{V\pi k_B T^{3/2}}\sqrt{E}\exp(-\frac{E}{k_B T})dE$ where $n(E)dE$ is the number of molecules per unit volume with energies between E and $E + dE$.

a) Show that the most probable energy of a molecule is $\frac{1}{2}k_B T$.
b) Calculate the ratio of the number of molecules per unit energy range having energy $k_B T$ to the number having 10 times this amount of energy.

B-4. The p-n junction has a current-voltage characteristic given by $I = I_0(\exp(\frac{eV}{k_B T}) - 1)$,

where the saturation current, I_0 = 1 nA. Calculate the resistance of the junction at 300 K for:

a) forward bias voltage of 0.5 V, and
b) a reverse bias voltage of 0.5 V.
c) Show that for <u>small</u> reverse bias voltages ($\ll k_B T / e$), the resistance is proportional to temperature, and
d) show that at 300 K it is about 26 MΩ. (Hint: $e^x = 1 + x + x^2/2! + \ldots$)

B-5. ^{228}Th (mass = 228.028750 u) decays to the ground state of ^{224}Ra (mass = 224.020218 u) with the emission of an α particle (mass = 4.002603 u).

a) What is the Q-value for the decay?

b) What is the kinetic energy of the α particle?

c) Decays also occur to an excited state of ^{224}Ra. If the α particles emitted in this process have a kinetic energy of 5.338 MeV, what is the energy of the accompanying photons?

B-6. Name any conservation law that is violated in each of the following hypothetical reactions:

a) $\pi^+ \rightarrow e^+ + \gamma$ b) $p + p \rightarrow p + \pi^+$ c) $\mu^- \rightarrow e^- + \bar{\nu}_e + \gamma$ d) $\Lambda^0 \rightarrow \pi^- + \pi^+$

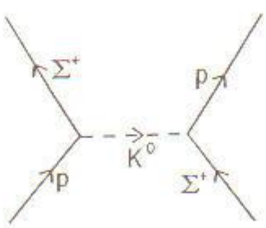
e) $K^+ \rightarrow \mu^+ + \bar{\nu}_\mu$

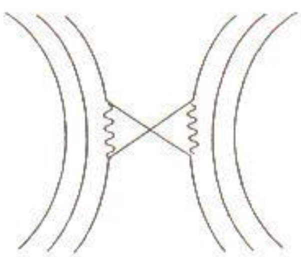
Supply the missing neutrino/antineutrino for each of the following processes:

f) $\pi^+ \rightarrow \mu^+ + ?$ g) $K^- \rightarrow \mu^- + ?$ h) $? + p \rightarrow n + e^+$ i) $? + n \rightarrow p + \mu^-$

j) $\tau^- \rightarrow \mu^- + ? + ?$

k) The diagram shows a proton and a Σ^+ interacting by exchange of a K^0 meson. Label the corresponding diagram on the right with the corresponding quark constituents. (The wavy lines are gluons. Colors are not needed)





Fall 03

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A-1.

BR A-2.

BR A-3.

A-4.

A-5.

BR B-1.

BR B-2.

BR B-3.

BR B-4.

A-1. Consider the reaction $^{14}N(\alpha, p)^{17}O$. (mass in amu: $m(^{14}N) = 14.003074$, $m(^{17}O) = 16.999132$, $m(^{18}F) = 18.000937$, $m(^{4}He) = 4.002602$, $m(^{1}H) = 1.007825$).

(a) Using the data above, calculate the energy released in the reaction (in MeV). Is the reaction exothermic or endothermic?

(b) Determine the threshold energy (in MeV) of the reaction, i.e., the minimum kinetic energy needed to initiate the reaction in the rest frame of the target ^{14}N.

(c) In the compound nucleus reaction model, the α capture results in the formation of an intermediate excited state of the ^{18}F nucleus, which subsequently decays to the outgoing proton and ^{17}O nucleus. Suppose that in a particular experiment, it is determined that the intermediate ^{18}F nucleus has excitation energy (energy above the ground state) 8.50MeV, as measured in the ^{18}F rest frame (the cm frame). Calculate the kinetic energy (in MeV) of the incoming α particle in this same reference frame.

A-2. Consider the decay of an unstable particle with rest mass energy $m_0c^2 = 600$ MeV into two other particles. Before the decay, the unstable particle moves horizontally with velocity $v_0 = 0.9c$. After the decay, the first decay product, which has rest mass energy $m_1c^2 = 250$ MeV, moves off with kinetic energy $K_1 = 120$ MeV at an angle $\theta_1 = 10°$ **above** the horizontal, while the second decay product moves off at an angle θ_2 **below** the horizontal.

(a) Find the total energy (in MeV) and momentum (in MeV / c) of the decaying particle. Note that this particle is relativistic.

(b) Determine the total energy (in MeV) of particle 2.

(c) Find the angle θ_2 and calculate the magnitude of the momentum (in MeV / c) of particle 2.

(d) Determine the mass of particle 2 (in MeV / c^2).

A-3. A highly energetic photon (wavelength λ, frequency ν) collides with an electron at rest. After the collision the angle θ between the photon's original direction $\vec{u}$ and its final direction $\vec{u}'$ is measured.

(a) Write down the conservation laws of energy and momentum. Treat the system as relativistic.

(b) Calculate the wave length shift $\Delta\lambda = \lambda' - \lambda$ in terms of the angle θ. Hint: start by writing an expression for the square of the electron's momentum $p^2 = \vec{p} \cdot \vec{p}$.

(c) Why, do you think, the result for $\Delta\lambda$ is remarkable?

A-4. In the laboratory frame of reference, an electron ($m_e c^2 = 0.511$ MeV) has kinetic energy $K = 1.2$ MeV.

(a) Determine the velocity (as a fraction of c), the momentum (in MeV / c) of the electron in the laboratory frame.

(b) Calculate the velocity (as a fraction of c), the momentum (in MeV / c) and the total energy (in MeV) of the electron in a frame of reference that moves along the electron's direction of motion with velocity $v = 0.5c$ with respect to the laboratory frame.

A-5. Consider a radioactive nuclear species (species 1) that decays with decay constant λ_1 (decays per second per nucleus). Suppose that the product of this decay (species 2) is also radioactive and decays with decay constant λ_2, $\lambda_1 < \lambda_2$. Define N_1 to be the number of nuclei of species 1 at time t and N_2 to be the number of nuclei of species 2 at time t. Suppose that we start with a pure sample of species 1; i.e., at time $t = 0$, let $N_1 = N_0$ and $N_2 = 0$.

(a) Write down the differential equations satisfied by N_1 and N_2 as function of time. Note that the rate equation for N_2 has competing terms due to the decay of species 1, which increases N_2, and the decay of species 2, which decreases N_2.

(b) Show that the solution to the rate equation obtained for N_2 with the proper initial condition is given by $N_2(t) = A[\exp(-\lambda_2 t) - \exp(-\lambda_1 t)]$ and determine the constant A.

B-1. Given the operators defined by $L_x = yp_z - zp_y$, $L_y = zp_x - xp_z$, and $L_z = xp_y - yp_x$, with $p_j = \frac{\hbar}{i}\frac{\partial}{\partial j}$, show that $[L_x, L_y] = i\hbar L_z$.

B-2. Consider the three-dimensional harmonic oscillator, which has a potential: $V(r) = \frac{1}{2}m\omega^2 r^2$. Use separation of variables in Cartesian coordinates ($r^2 = x^2 + y^2 + z^2$), and $\psi(x,y,z) = X(x)Y(y)Z(z)$ and determine the general expression for the bound state energy.

B-3. A particle in an infinite square well potential has the initial-state wave function $\Psi(x,0) = A_1\psi_1(x) + A_2\psi_2(x)$ where $A_1 = 2A_2$.

(a) Normalize the $\Psi(x,0)$.

(b) Write the full expression for $\Psi(x,t)$.

(c) Calculate $\langle H \rangle$. Recall that $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$.

B-4. Two identical bosons are placed in an infinite one-dimensional square well of width a.

(a) Write the wave function and energies for the ground state and the first excited state. Be sure to use the proper symmeterization.

(b) Now let's say that the bosons interact *weakly* through the potential $V(x_1, x_2) = -aV_0\delta(x_1 - x_2)$, where V_0 is a constant with dimensions of energy. Find the first-order correction to the ground state energy.

(c) Find the first-order correction to the first excited state energy.

Fall 04

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A-1.

A-2.

A-3.

A-4.

A-5.

B-1.

B-2.

B-3.

B-4.

A-1. In an inertial system, S, a person at the origin holds a laser pointing along the positive y axis. Relative to system S, a fast space-ship is moving at c/2 in the negative x direction. An astronaut in the space-ship measures the velocity of the light beam arriving from the laser at the instant when observers in S see the space-ship pass the positive y axis. According to the astronaut, what is the speed and direction of the light beam?

A-2. Antiprotons $\overline{p}$ were first produced in the lab by projecting high-energy protons towards protons at rest. The simplest reaction is:

$$p + p \rightarrow p + p + (p + \overline{p})$$

What is the minimum kinetic energy of the incident proton necessary to produce this reaction? (Hint: The minimum kinetic energy occurs when all outgoing particles are at rest relative to the center of mass.)

A-3. Consider a gas of hydrogen atoms. At what temperature will 99% of the atoms be in the ground state (n = 1), the remainder essentially being in the first excited state (n = 2)? (Note that the degeneracy of the hydrogen atom is $2n^2$.)

A-4. One can assume that, at the time of the supernova explosion that subsequently gave birth to our solar system, the two isotopes ^{235}U and ^{238}U were present in equal amounts. At the present time, the relative abundance of ^{235}U is 0.7% and of ^{238}U, 99.3%. ^{235}U has a half-life of 2.25×10^{16} s and ^{238}U has a half-life of 1.42×10^{17}s. Using this information compute the time of the supernova explosion.

A-5. Consider the Earth-sun system as the gravitational analog to the hydrogen atom.

a) Write down the potential energy function using m for the earth mass and M as the solar mass.

b) What is the "Bohr radius" for the system? Calculate the numerical value.

c) Equate the classical energy of a planet in circular orbit of radius r_0 to the "Bohr energy". Show that $n = \sqrt{r_0 / a}$ and determine the earth's quantum number, n.

d) If the earth made a transition to the next lower level $(n-1)$, how much energy would be released (in Joules). What is the wavelength of the graviton?

B-1. Given that

$$L_{\pm} = L_x \pm iL_y$$

and that

$$L_{\pm}\left|lm\right\rangle = \hbar\sqrt{l(l+1) - m(m\pm1)}\left|l(m\pm1)\right\rangle.$$

a) Show that $\left\langle L_x^2 \right\rangle = \left\langle L_y^2 \right\rangle$.

b) Determine $\left\langle L_x^2 \right\rangle$ for and arbitrary state

B-2. Consider an infinite cubic well:

$$V(x,y,z) = \begin{cases} 0 & if\ 0 \le x \le a, 0 \le y \le a, 0 \le z \le a \\ \infty & elsewhere \end{cases}$$

a) Determine the stationary-state wave functions and energies.
b) If we now include a perturbation of

$$H' = V_0 a^3 \delta(x - \frac{a}{4})\delta(y - \frac{3a}{4})\delta(z - \frac{a}{4}),$$

find the first order correction to the ground state energy.

B-3. For three dimensional potentials we can separate the radial components of momentum p_r and position r. In operator form, these act on the radial part of the wave function $R(r)$ according to

$$\hat{p}_r R(r) = -i\hbar\frac{1}{r}\frac{d}{dr}(rR(r)) \text{ and } \hat{r}R(r) = rR(r)$$

a) Find the commutator $[\hat{p}_r, \hat{r}]$.

b) Use the above commutator to determine $\Delta p_r \Delta r$. This is just the uncertainty relation for p_r and r.

B-4. Consider a particle of mass m that is moving in a three dimensional potential

$$V(x,y,z) = \begin{cases} \frac{1}{2}m\omega^2 z^2 & for\ 0 \le x \le y, 0 \le y \le a \\ \infty & elsewhere \end{cases}$$

a) Determine the n^{th} stationary state wave function and energy for the particle.

b) For $\hbar\omega > 3\pi^2\hbar^2 / ma^2$, determine the energies for the ground state(s) and the first excited state(s). What are the degeneracies for these states?