Weighted Mean or Expected value

Example: "Would You Pay $5 to Play?"

A game works like this: You draw one card from a bag containing **10 cards**.

- 6 cards say **"Win $2"**
- 3 cards say **"Win $8"**
- 1 card says **"Win $30"**

What is the **expected amount you win per game**?

X	P(x)	X * P(x)
$2	0.60	$1.20
$8	0.30	$2.40
$30	0.10	$3.00
	1.00	**$6.60**

So the expected value is E(X) = 6.60

1. **The casino charges $5 to play. Would you play?**

Since the expected winnings are **$6.60**, the expected **profit** to the player is $6.60−$5.00 = $1.60

This game favors the player.

2. **What if the casino charges $8 to play?**

$6.60 − $8.00 = −$1.40

The player loses an average of **$1.40 per game in the long run**.

Expected value does NOT mean you should expect to win $6.60 on your next game. You can never even win $6.60!

It means that if thousands of people play, their **average winnings per play should approach $6.60**.

Harmonic mean is the **average speed over equal distances**.

Example: A trip to campus and back

Suppose a student drives:

- **30 miles to campus** at **30 mph**
- **30 miles back home** at **60 mph**

You might think the average speed is: $\bar{x} = (30+60)/2 = 45$ mph

But that's **not correct**, because the student spends more time driving at the slower speed.

Since the distances are equal, use the **harmonic mean**:

$$H = \frac{2}{\frac{2}{60} + \frac{1}{60}} = \frac{2}{3/60} = \boxed{40\ \text{mph}}$$

Why 40 mph makes sense

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}} = \frac{60}{1.5} = \boxed{40\ \text{mph}}$$

Teaching rule: When averaging **rates** (speed, productivity, price per unit, etc.) over **equal quantities**, the harmonic mean often appears.

Geometric mean, percentages must first be written as **100% + the growth rate**.

💰 Example: Average annual investment return

Suppose an investment earns:

- **Year 1: +20%**
- **Year 2: +80%**

What is the average annual percentage return?

To calculate the arithmetic mean: $\bar{x} = (20\% + 80\%) / 2 = 50\%$

But because investment returns **compound**, we use the **geometric mean**:

$$GM = \sqrt{(100\% + 20\%) \cdot (100\% + 80\%)} - 100\% = \mathbf{47\%}$$

So the **average annual return is about 47%**, not 50%.

Why do we add 100%?

Because after a 20% increase, you have **120% of the original amount**, and after an 80% increase, you multiply that result by **180%**.

For example, starting with \$100:

$\$100 \cdot (1.20) \cdot (1.80) = \$216$

Using the geometric mean:

$\$100 \cdot (1.47^2) = \$216$

So **47% per year for two years produces the same final result**.

Quadratic mean is useful when values can be **positive and negative**, but we care about their **magnitude**.

🌡 **Example: Temperature deviations**

Suppose during four days the temperature differs from the forecast by:

$+2°, -2°, +4°, -4°$

If we calculate the ordinary arithmetic mean:

The result: $\bar{x} = 0$

That could misleadingly suggest that there was **no forecasting error**.

Instead, calculate the **quadratic mean**:

$$Q = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}$$

So the **typical magnitude of the forecasting error is about 3.16°**.

Mean	When useful	Simple example
Arithmetic	Ordinary quantities	Average exam grade
Geometric	Compounding	Average investment growth
Harmonic	Rates	Average speed
Quadratic	Magnitudes of ± values	Forecasting errors