Intermediate Microeconomics — Week 4

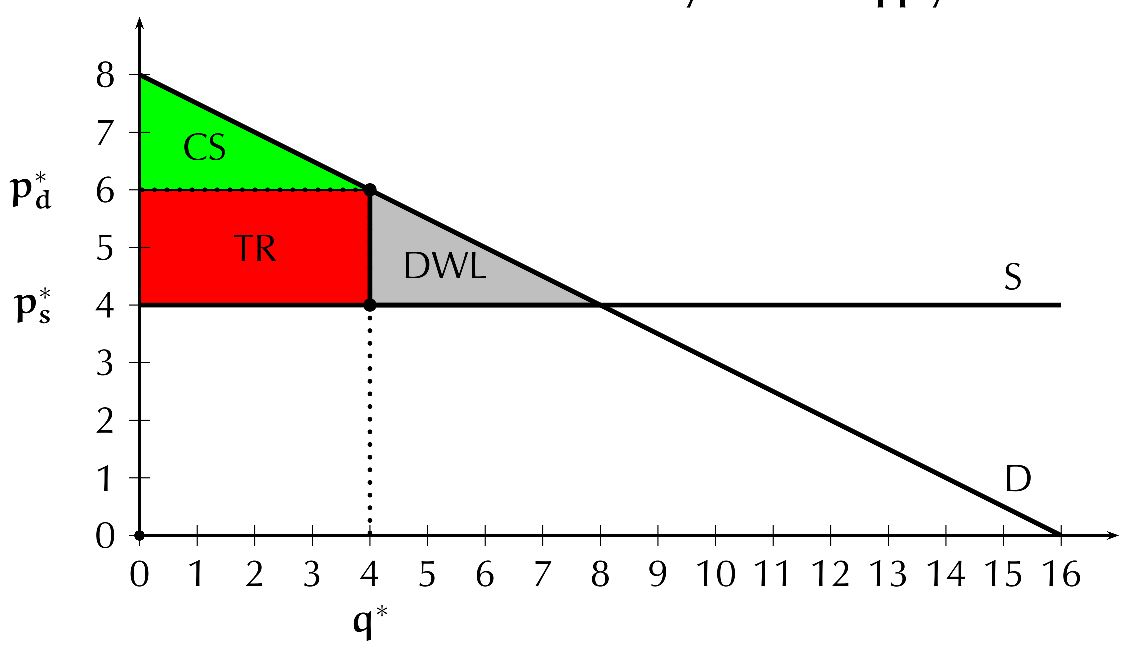
Professor Boyd **Sept. 13 & 15, 2022**

3.6.13 Taxation with Perfectly Elastic Supply

The elasticity of supply and demand have a considerable impact on the welfare effects of taxes. When supply is perfectly elastic (horizontal), the equilibrium supply price is not affected by the \$2 tax. As a result, 100% of the tax falls on the buyer, as does 100% of the excess burden.

Due to the perfectly elastic supply, the producer's surplus is zero both with and without the tax.

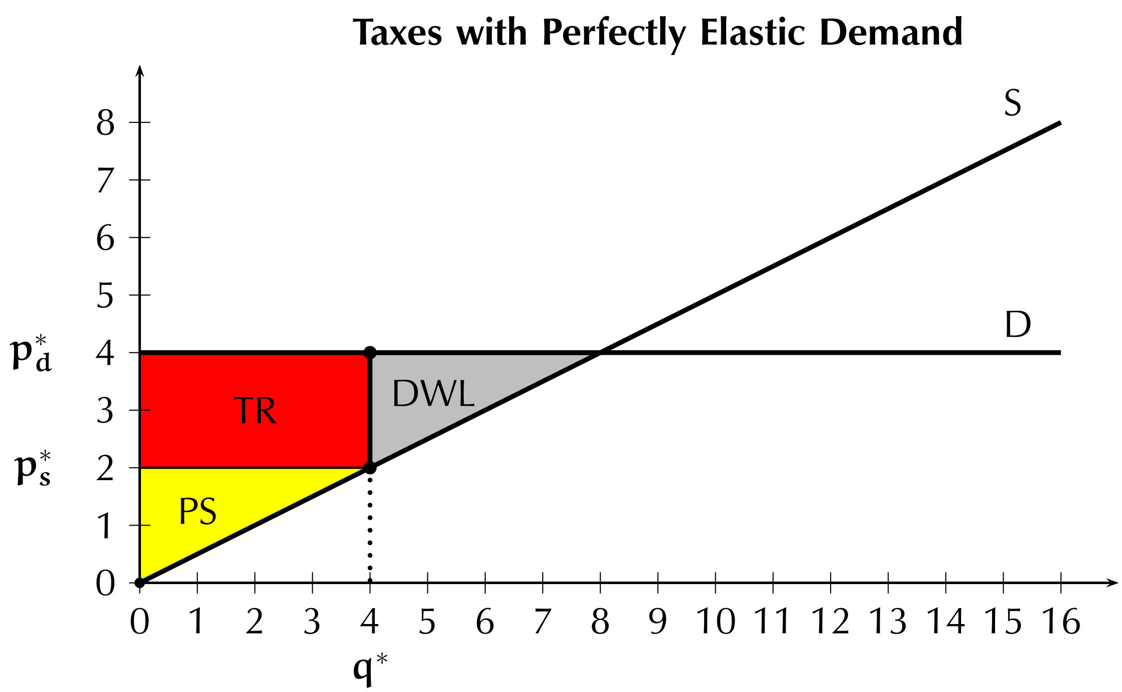
Taxes with Perfectly Elastic Supply



3.6.14 Taxation with Perfectly Elastic Demand

When demand is perfectly elastic (horizontal), the equilibrium demand price is not affected by the $2 tax. As a result, 100% of the tax falls on the seller, as does 100% of the excess burden.

The consumer's surplus is zero with and without the tax, due to perfect elasticity, this time of demand.



3.6.15 The Burden of Taxation

One useful way to examine the burden of taxation is to look at its effect on both the buyer's and seller's prices. We compare the after-tax prices to the equilibrium price without a tax. As show in the table below, the no-tax price was $p^* = \$4$, while with the tax, $p_d^* = \$6$ and $p_s^* = \$3$. The difference in price is $p_d^* - p^* = \$6 - \$4 = \$2$ for the buyers, and $p^* - p_s^* = \$4 - \$3 = \$1$ for the sellers.

Price	**No Tax**	**With Tax**	**Change**
p_d^*	\$4	\$6	+\$2
p_s^*	\$4	\$3	-\$1

This allows us to divide both the direct burden of the tax, the tax paid, between the buyers and sellers. Notice that $\$2 + \$1 = \$3$, the entire tax rate. It follows that the buyers pay $\$2 \times q^* = \$2 \times 4 = \$8$, and the sellers pay $\$1 \times 4 = \$4$. In this example, two-thirds of the tax is paid by the buyers.

3.6.16 Dividing the Tax

More generally, the buyer's portion of an excise or sales tax is

$$t_d = p_d^* - p^*$$

where p^* is the no-tax equilibrium price and p_d^* is the after-tax buyer's price. Similarly, the seller's portion of the tax is

$$t_s = p^* - p_s^*$$

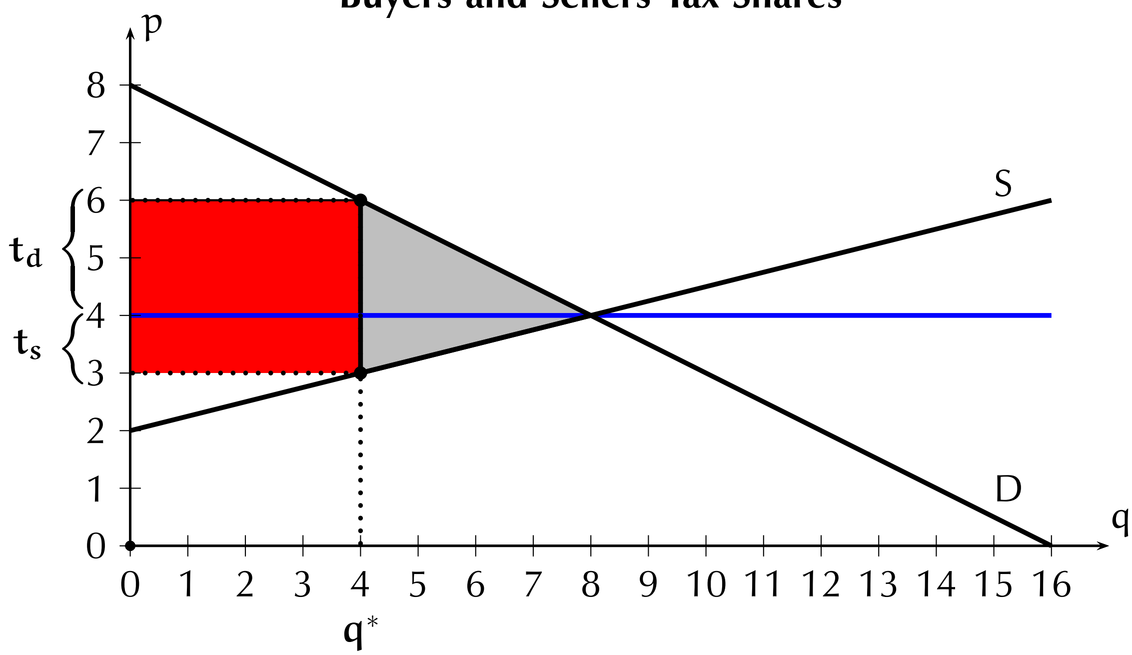
where p_s^* the seller's after-tax price.

Now

$$\begin{aligned} t_s + t_d &= (p_d^* - p^*) + (p^* - p_s^*) \\ &= p_d^* - p_s^* = t, \end{aligned}$$

showing that the buyer's and seller's portions add to the entire tax.

Buyers and Sellers Tax Shares



3.6.17 Tax Shares

To study this in more detail, we define the buyer's share of the tax to be $s = t_b/t$. The seller's share is then $(1 - s)$ because

$$\frac{t_s}{t} = \frac{t - t_b}{t} = 1 - \frac{t_b}{t} = 1 - s.$$

We can now compute the shares in terms of the elasticities of supply and demand.

Recall the definitions of elasticities of demand and supply:

$$E_d = \frac{\%\Delta q_d}{\%\Delta p_d} \qquad \text{and} \qquad E_s = \frac{\%\Delta q_s}{\%\Delta p_s}$$

Compared with equilibrium, in the tax equilibrium, both the quantity supplied and quantity demanded must be reduced by the same amount. This ensures the market still clears. The percentage change is also the same since the quantity reduction is from the same starting point.

3.6.18 Computing the Tax Shares I

The quantity reductions are

$$\%\Delta q_d = E_d \%\Delta p_d \qquad \text{and} \qquad \%\Delta q_s = E_s \%\Delta p_s.$$

Since the quantity reductions are the same, we can equate the elasticity terms, obtaining

$$E_d \%\Delta p_d = E_s \%\Delta p_s. \tag{3.6.1}$$

Now

$$\%\Delta p_d = \frac{t_d}{t} = \frac{st}{t} = s$$

and

$$-\%\Delta p_s = \frac{t_s}{t} = \frac{(1-s)t}{t} = 1 - s.$$

The last equation has a minus sign since because the seller's after-tax price is lower than the equilibrium price, while the buyer's after-tax price is higher than the equilibrium price.

Substituting these expressions in equation (3.6.1), we obtain

$$-sE_d = (1-s)E_s. \tag{3.6.2}$$

3.6.19 Computing the Tax Shares II

We start with equation (3.6.2),

$$-sE_d = (1-s)E_s. \tag{3.6.2}$$

Then the tax share obeys

$$s\big(E_s - E_d\big) = E_s \qquad \text{and} \qquad s\big(|E_s| + |E_d|\big) = |E_s|$$

where the second equation uses the fact that elasticity of demand is negative. The buyer's tax share is then

$$s = \frac{|E_s|}{|E_s| + |E_d|}$$

and the seller's tax share is

$$1 - s = \frac{|E_d|}{|E_s| + |E_d|}$$

3.6.20 Some Example Tax Shares

1. Suppose $E_d = -2$ and $E_s = +3$, so supply is more elastic than demand. Then the buyers tax share is

$$s = \frac{|E_s|}{|E_d| + |E_s|} = \frac{3}{5} = .6.$$

The buyers pay 60% of the tax, and the sellers pay 40%.

2. Suppose $E_d = -3$ and $E_s = +1.5$, so demand is more elastic than supply. Then the buyers tax share is

$$s = \frac{|E_s|}{|E_d| + |E_s|} = \frac{1.5}{4.5} = \frac{1}{3}.$$

The buyers pay $\frac{1}{3}$ of the tax, and the sellers pay $\frac{2}{3}$.

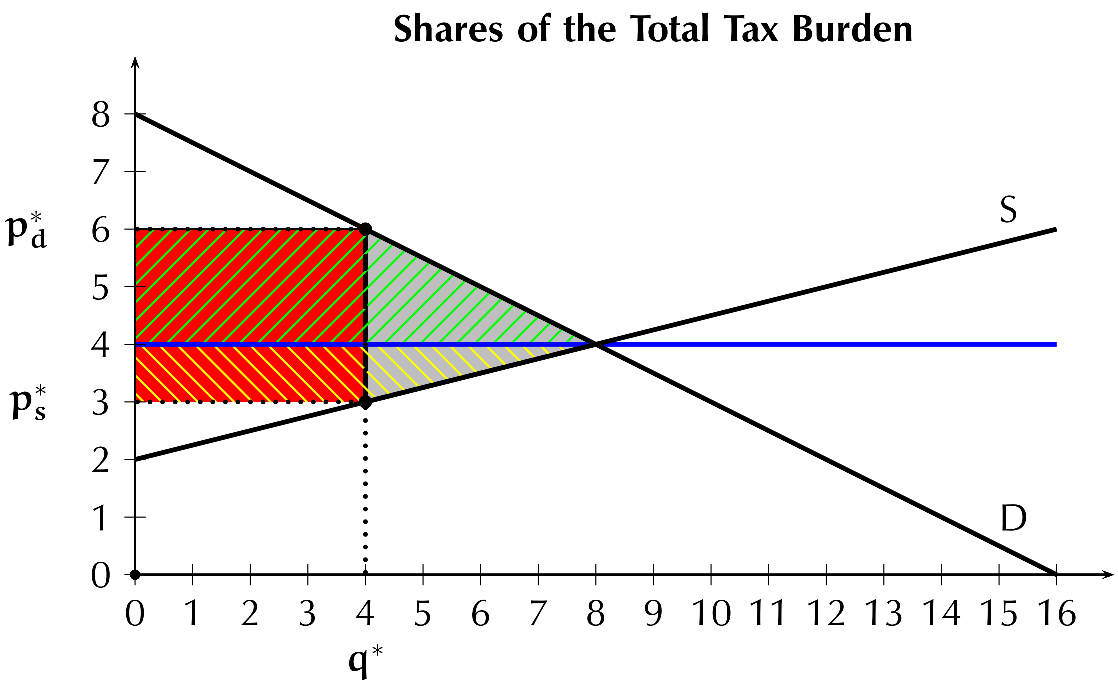
3. Suppose $E_d = -0.2$ and $E_s = 0.1$. Again, demand is more elastic than supply. The buyers tax share is

$$s = \frac{|E_s|}{|E_d| + |E_s|} = \frac{0.1}{0.3} = \frac{1}{3}.$$

The buyers pay $\frac{1}{3}$ of the tax, and the sellers pay $\frac{2}{3}$.

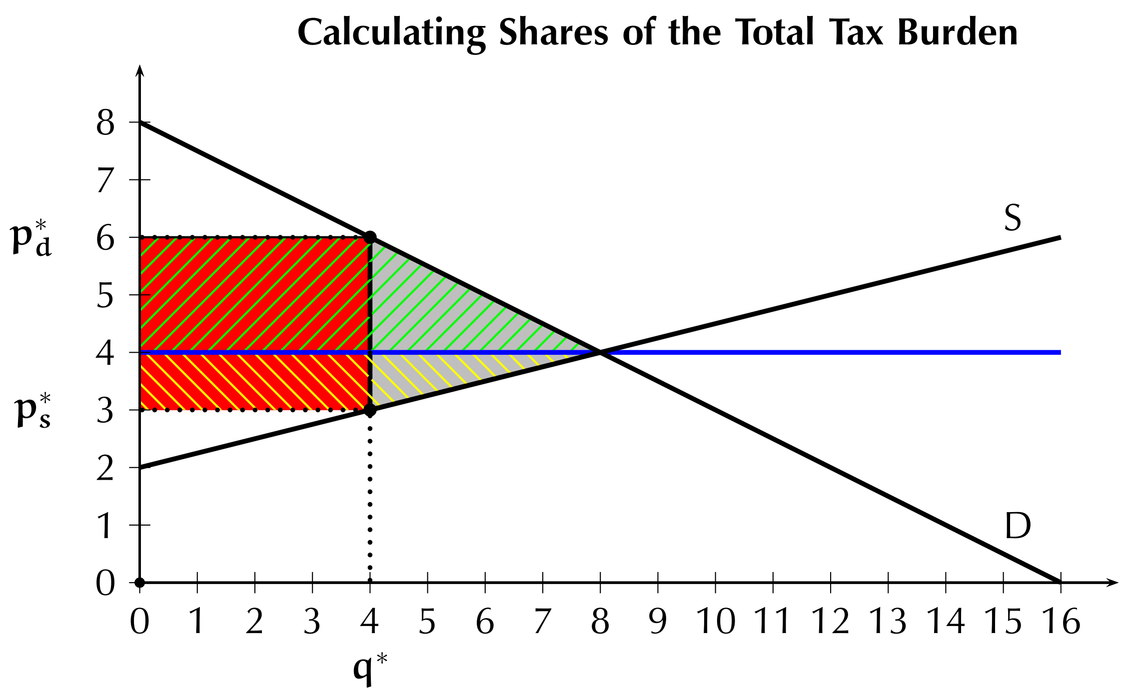
3.6.21 Shares of the Tax Burden I

The blue line at $p = \$4$ indicates the no-tax equilibrium. The direct burden of the tax is the red area. The gray area is the excess burden. The buyer's burdens (direct and excess) are the above the blue line, with green hatching. The seller's burdens are below the blue line, with yellow hatching. In both cases, the burdens are proportional to the change in price faced by the buyers and sellers, respectively.



3.6.22 Shares of the Tax Burden II

Here the buyers pay $\frac{2}{3}$ of the tax, and bear $\frac{2}{3}$ of the burden, a 1:2 ratio. We can calculate the numbers off the diagram. The buyers' price is increased by \$2, and the sellers' price reduced by \$1. The buyers pay \$8 in tax, the sellers pay \$4, again a 1:2 ratio. Finally, the excess burden triangle lost by the buyers is \$4, while the excess burden triangle for the sellers is \$2—also a 1:2 ratio.

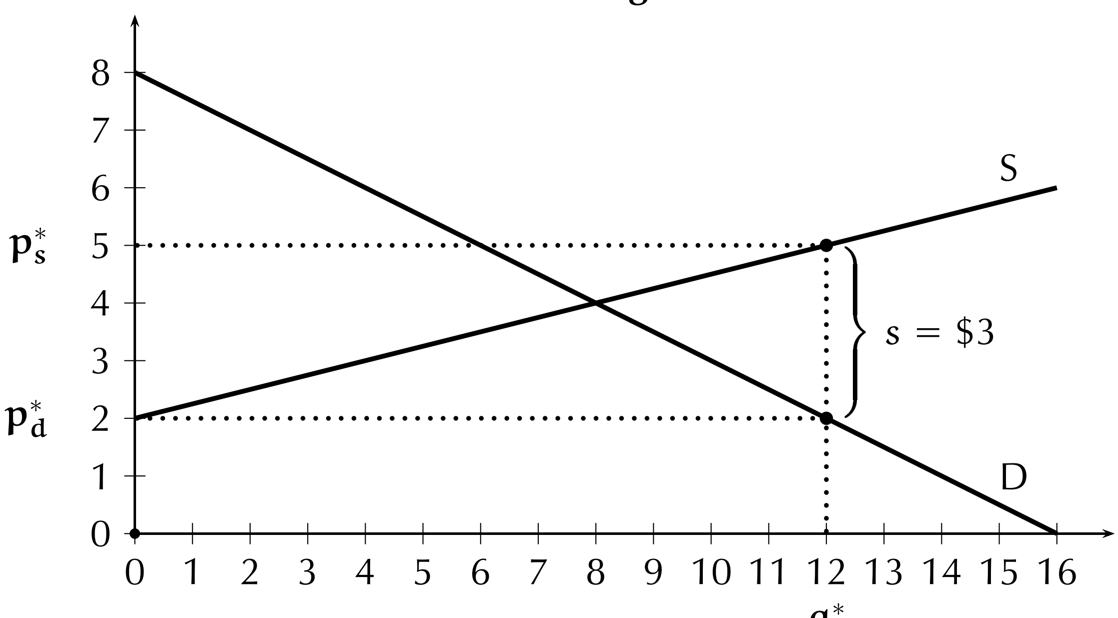


3.6.23 Subsidies

Per-unit subsidies can be treated like negative taxes. Instead of $p_d = p_s + t$, where $t > 0$ is the per unit tax, subsidies are defined by $p_d = p_s - s$, where $s > 0$ is per unit subsidy. On supply-demand diagrams, subsidies show up to the right of the no-tax equilibrium, whereas taxes are placed to the left. Notice that $p_s^* > p_d^*$.

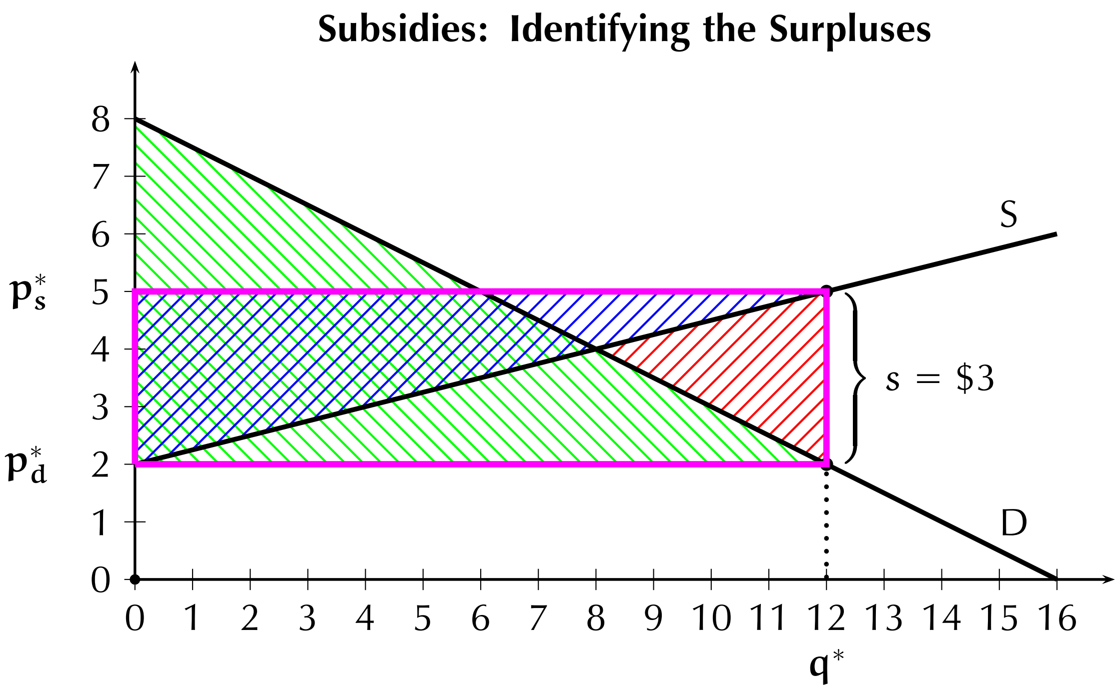
The government pays the subsidy, which costs the government $s \times q^* = (p_s^* - p_s^*) \times q^*$.

Subsidies as Negative Taxes



3.6.24 Subsidies: Surpluses and Losses I

The welfare analysis is rather more complex for subsidies. The green area is the consumer's surplus. The blue area is producer's surplus. The cross-hatched area gets counted twice, for both consumers and producers surplus. The magenta box is the cost to the government, which must be subtracted when computing the total surplus. The red hatched area to the right is the deadweight loss.



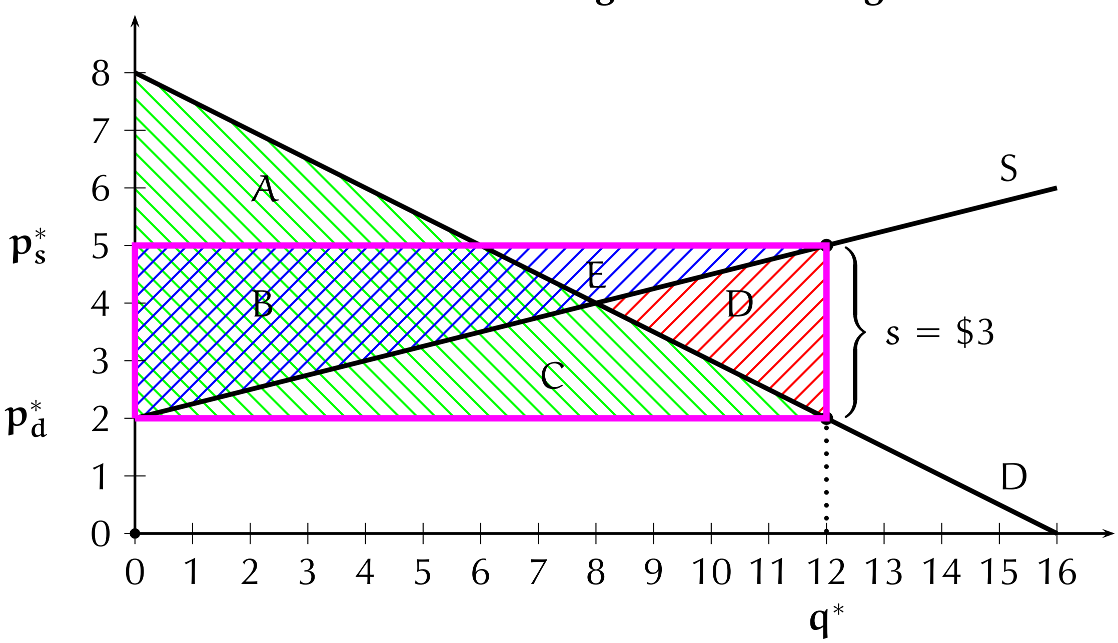
3.6.25 Subsidies: Surpluses and Losses II

Here's another way to identify the total surplus.

$$\begin{aligned} CS &= A + B + C \\ PS &= B + E \\ \text{Subsidy Cost} &= B + C + D + E \\ TS &= CS + PS - \text{Subsidy Cost} = A + B - D \end{aligned}$$

Area D is subtracted from the maximum possible surplus $A + B$ to find the total surplus with tax. That makes D the deadweight loss.

Subsidies: Triangles and Rectangles



4.1 Chapter 4: Consumer Theory

When spending your paycheck, your money is used to purchase a variety of goods and services. It pays the rent and utilities, buys various food and beverage items, clothing, gasoline, sometimes a new car, etc. The totally of what you buy is called a **market basket** or **bundle** of goods. When allocating the money from your paycheck, you need to take all of these options into account.

Consumers choose bundles or market baskets of goods.

4.1.1 The Two Pillars of Consumer Choice

To analyze consumer choice, we focus on opportunities and preferences.

1. Consumer choice is **constrained by their resources**, by their opportunities for consumption. More to the point, it is limited **by what consumers can afford** (budget constraint).
2. Consumers **pick the best bundle they can afford** (utility maximization).

4.1.2 Basic Properties of Consumer Preference

Consumers choose bundles or market baskets of goods. They base these choices on their **preferences**, meaning their ranking of the possible consumption bundles. We will focus on four properties of preferences.

1. **Completeness:** Consumers can compare all bundles they can afford.
2. **Monotonicity:** More of all goods is better (or at least no worse).
3. **Transitivity:** If a consumer chooses between three bundles, A, B, and C, and prefers bundle A to bundle B, and also prefers bundle B to bundle C, then that consumer also prefers bundle A to bundle C.
4. **Diminishing Subjective Value:** The more a consumer has of a particular good, the less in other goods they are willing to give up to get one more unit of that good.

4.1.3 Paradox of Voting

Although we assume consumer choice obeys transitivity, there are methods of choosing alternatives that are not transitive. One is voting.

Suppose there are three voters: Ada, Bill, and Chris. These voters must choose one of three possible proposals, which I'll call A, B, and C. They rank the proposals as follows:

	Ada	Bill	Chris
#1	A	B	C
#2	B	C	A
#3	C	A	B

In a vote between proposals A and B, Ada and Chris favor A, while Bill prefers B. The vote is 2-1 for A.

Between B and C, Ada and Bill vote for B, while Chris votes for C. Then B wins, 2-1.

Finally, between C and A, Ada favors A, while Bill and Chris prefer C. So C wins, 2-1.

Ranking by voting is intransitive. Option A is beats B which beats C. If it were transitive, we could conclude A beats C. But this is false as A loses to C, 2-1.

This observation was originally made by the Marquis de Condorcet (1743–1794) in 1785.[1] It was independently discovered by Lewis Carroll (Charles L. Dodgson) in the late 18$^{\text{th}}$ century. Its full significance for voting theory was realized by Duncan Black in a series of papers in the 1940's.

[1] Marquis de Condorcet (1785). *Essai sur l'application de l'analyse à' la probabilité des décisions rendues à la pluralité des voix*.

4.1.4 Choice without Transitivity

It is possible to study consumer choice under substantially weaker assumptions. In fact, **none** of our assumptions are absolutely necessary.

However, our assumptions are intuitive, they often hold, and it is **much easier** to study consumer choice using them.

4.1.5 Preference Rankings: Utility

The term **utility** is used to refer to the intensity of preference for consumption bundles. It applies to consumption **bundles**, not individual **goods**.

We often use a numerical scale to express intensity of preference, if so, it is also called utility.

This numerical utility scale is arbitrary. Doubling it, squaring it, adding 42 to it, and taking its exponential all leave the relative ranking of bundles unchanged and all those utility scales are considered equivalent. Any change of the scale that doesn't affect the rankings of the bundles is allowed. In other words, any increasing function of a utility scale gives us a new utility scale describing the same preferences.

This arbitrary aspect of utility means we shouldn't compare utility across individuals. This doesn't stop people from doing so, but such comparisons are always completely arbitrary from a moral point of view.

4.1.6 How the Utility Scale Works

Suppose there are two goods: apples (a) and gummy bears (b).

One consumer has preferences described by the utility function $u(a, b) = a \times b$. We write a consumption bundle as (a, b) meaning a apples and b gummy bears. Thus $(2, 1)$ is 2 apples and 1 bear.

As far as our consumer is concerned, the bundles $(2, 1)$ and $(1, 2)$ have identical utility (subjective value) because

$$u(2, 1) = 2 \times 1 = 2 = 1 \times 2 = u(1, 2).$$

The average bundle, $(1.5, 1.5)$ is a better choice than either because

$$u(1.5, 1.5) = 1.5 \times 1.5 = 2.25 > 2.$$

We don't know whether it's a little better or a lot better, only that it is better.

This particular utility is a type of Cobb-Douglas utility. The function can also be used to study production. Charles Cobb and Paul Douglas used it this way in the interwar period.[2] That's why their names are on it. Douglas later became a US Senator from Illinois from 1949–1967 where he supported both fiscal restraint and civil rights. Martin Luther King, Jr. called him the "greatest of all the Senators."

[2] Their first relevant paper was Charles W. Cobb and Paul H. Douglas (1928), A theory of production, *Amer. Econ. Rev.* **18**, no. 1. Supplement, Papers and Proceedings, 139–165.

4.1.7 A Second Consumer: Different Preferences and Utility

A second consumer's preferences are summed up by the utility function $v(a, b) = b + \sqrt{a}$. Then $v(1, 2) = 2+1 = 3$, but $v(2, 1) = 1+\sqrt{2} = 2.41$. For **this** consumer, $(2, 1)$ is better than $(1, 2)$, while our first consumer ranked them as equal.

What about $(1.5, 1.5)$? Now $v(1.5, 1.5) = 1.5 + \sqrt{1.5} = 2.72$, so $(1.5, 1.5)$ is better than $(2, 1)$, but worse than $(1, 2)$. Once again, we don't know how much better or worse any of the bundles are. Utility only tells us the relative ranking.

4.1.8 Utility and Preferences

We need to consider whether such utility scales can actually describe preferences. Indeed they do!

If we have two consumption bundles (a, b) and (a', b'), we can compare $u(a, b)$ and $u(a', b')$ to find out which is preferred (or if it's a tie). That means our ranking via utility is **complete**.

If $a \geq a'$ and $b \geq b'$, then

$$u(a, b) = a \cdot b \geq a' \cdot b' = u(a', b')$$

showing that our ranking is **monotonic**.

Now suppose $u(a_1, b_1) \geq u(a_2, b_2)$ and $u(a_2, b_2) \geq u(a_3, b_3)$, meaning that bundle (a_1, b_1) is utility preferred to (a_2, b_2) and bundle (a_2, b_2) is utility preferred to (a_3, b_3). Then

$$u(a_1, b_1) \geq u(a_2, b_2) \geq u(a_3, b_3),$$

so (a_1, b_1) is utility preferred to (a_3, b_3), showing the utility ranking is **transitive**.

That leaves diminishing subjective value, which will have to wait a bit until we learn more about it.

4.1.9 From Preferences to Utility

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Can we go the other way, from preferences to a utility scale? Generally speaking, we can. If the preferences also are continuous in the sense that bundles that are numerically close to each other are close in terms of preference, and if the preferences are also complete and transitive, then there are mathematical theorems that allow us to describe those preferences via a utility function.

This involves some advanced math (e.g., topology) and we won't attempt to show this here. It is enough to know that it is generally possible.[3]

[3] If you insist on a reference, try Gerard Debreu (1954) Representation of a preference order by a numerical function, in Decision Processes (R. M. Thrall, C. H. Coombs, and R. L. Davis, eds.), Wiley, New York, or Gerard Debreu (1960) Topological methods in cardinal utility theory, in Mathematical Methods in the Social Sciences, 1959 (Kenneth J. Arrow, Samuel Karlin, and Patrick Suppes, eds.), Stanford University Press, Stanford. Debreu (1921–2004) won the 1983 Nobel Memorial Prize in Economic Sciences.

4.1.10 Arbitrariness of the Utility Scale

Many utility scales describe the same preferences. In particular, if you take an increasing transformation of a utility scale, it still describes the same preferences. Such utility scales are called **equivalent**.

For example, the functions $f(x) = 2x$, $f(x) = x^3$ and $f(x) = \ln x$ are all increasing functions. It we use $u(a, b) = a \times b$, the transformations to $2a \times b$, a^3b^3, or $\ln a + \ln b$ all describe the same preferences.

In principle, we can infer preferences from economic behavior. But there is no way to infer a particular utility scale from behavior. Any increasing transformation gives us the same scale, and describes the same economic behavior. That's why any particular utility scale is arbitrary. You can pick whatever equivalent utility scale that is most convenient for your economic problem.

4.2 Marginal Utility

A lot of economic analysis is based on marginal changes and their kin, derivatives. This applies to utility too. Although changes in utility also suffer from arbitrariness, **their ratios have economic meaning**. Computing the changes in utility, marginal utility, is the first step toward finding these ratios.

The **marginal utility** of a change in consumption is the rate of change in utility itself. We usually call the units of utility "utils", so marginal utility is measured in utils per consumption unit. For milk, it might be in utils per gallon or utils per quart. For meat, utils per ounce or utils per pound, etc.

4.2.1 Calculating Marginal Utility

Let's consider a Cobb-Douglas utility function of the form $u(a, b) = a \times b$. Suppose consumption of apples changes from a to $a + \Delta a$, while the consumption of gummy bears remains at b. The **marginal utility of apples** is the rate of change of utility as we move from the consumption bundle (a, b) to $(a + \Delta a, b)$. This is measured in utils per apple. Typically, this number will depend on the bundle (a, b).

We compute the marginal utility of apples, MU_a, as follows:

$$
\begin{aligned}
MU_a(a, b) &= \frac{\Delta u}{\Delta a} \\
&= \frac{u(a + \Delta a, b) - u(a, b)}{\Delta a} \\
&= \frac{(a + \Delta a)b - ab}{\Delta a} \\
&= \frac{ab + (\Delta a)b - ab}{\Delta a} \\
&= \frac{(\Delta a)b}{\Delta a} \\
&= b.
\end{aligned}
$$

4.2.2 More Marginal Utility

There's also a marginal utility of gummy bears, MU_b. The definition is similar, and for our utility function $u(a, b) = a \times b$, $MU_b(a, b) = a$.

So at $(3, 2)$, we find that $MU_a(3, 2) = 2$ and $MU_b(3, 2) = 3$. As mentioned before, marginal utility depends on our choice of a bundle (a, b). In this case, $MU_a(1, 7) = 7$ and $MU_b(1, 7) = 1$.

4.2.3 Marginal Utility with Utility v, I

Let's do the same thing for $v(a, b) = b + \sqrt{a}$. However, the square root means we need calculus to compute the marginal utility properly. The idea is that the marginal utility MU_a tells us the slope of the utility graph with respect to a. To calculate that, we use the slope of the tangent, which is given by the derivative.

Now there's a slight complication. There are two variables, and we're only paying attention to one of them. The correct derivative to use is the **partial derivative**, $\partial u/\partial a$. To calculate it, we pretend that a is the only variable and that b is just some constant.

$$\begin{aligned} MU_a &= \frac{\partial v}{\partial a} = \frac{\partial b}{\partial a} + \frac{\partial \sqrt{a}}{\partial a} \\ &= 0 + \frac{\partial a^{1/2}}{\partial a} = \frac{1}{2} a^{-1/2} \\ &= \frac{1}{2\sqrt{a}} \end{aligned}$$

by the power law for derivatives, applied to $\sqrt{a} = a^{1/2}$.

4.2.4 Marginal Utility with Utility v, II

Similarly,

$$\begin{aligned} MU_b &= \frac{\partial v}{\partial b} = \frac{\partial b}{\partial b} + \frac{\partial \sqrt{a}}{\partial b} \\ &= 1 + 0 = 1. \end{aligned}$$

As a reminder, the units for MU_a are utils per apple; for MU_b, they are utils per gummy bear.

4.3 Indifference Curves

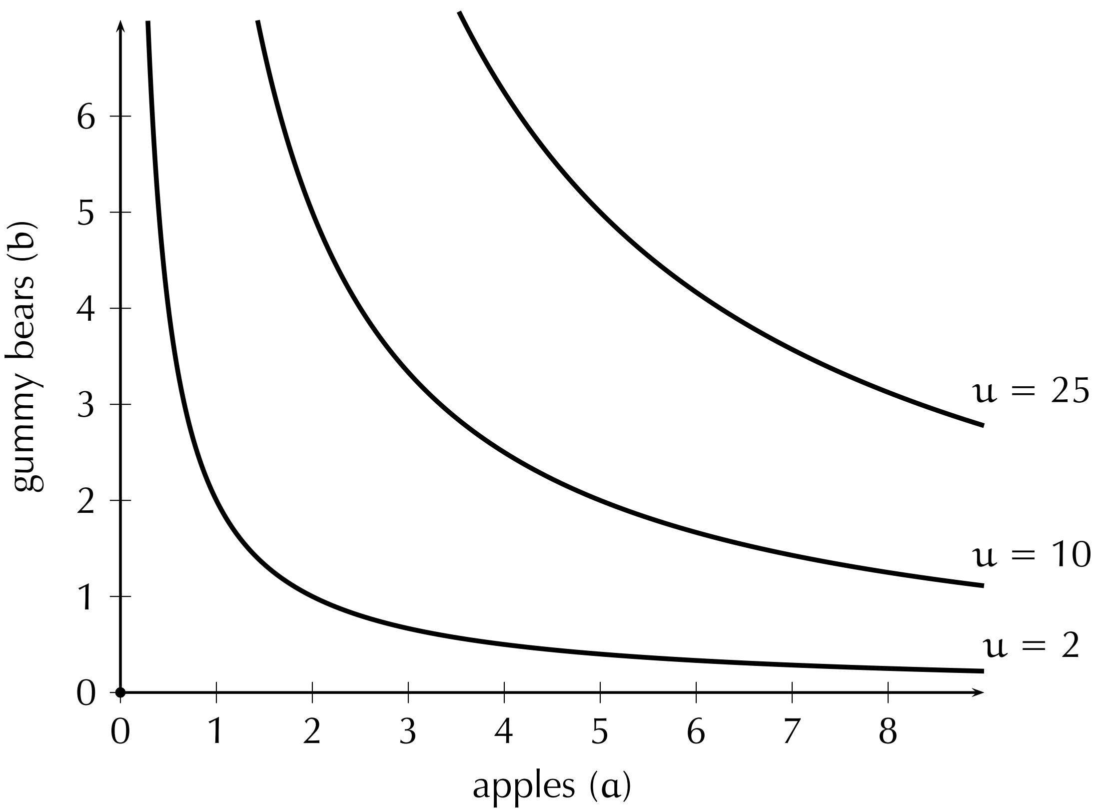
Of course, there is a way to graph utility. We don't usually graph it directly as a function. Even with two goods, that would require a three-dimensional graph. In real world cases, where they are literally millions of goods to consider, interpreting the graph would be impossible.

Fortunately, the principles underlying consumer behavior become clear with two goods. We simplify the graph by graphing the level sets, the sets of points that all have the same utility. In economics, we refer to these as **indifference curves**. The point is that if utility of two bundles is the same, the consumer ranks those as equally good. They don't care which bundle they get. It is all the same to the consumer.

4.3.1 Three Indifference Curves

The figure below includes three of the indifference curves for the Cobb-Douglas utility function $u(a, b) = ab$. The equations for them are $u(a, b) = 2$, $u(a, b) = 10$, and $u(a, b) = 25$.

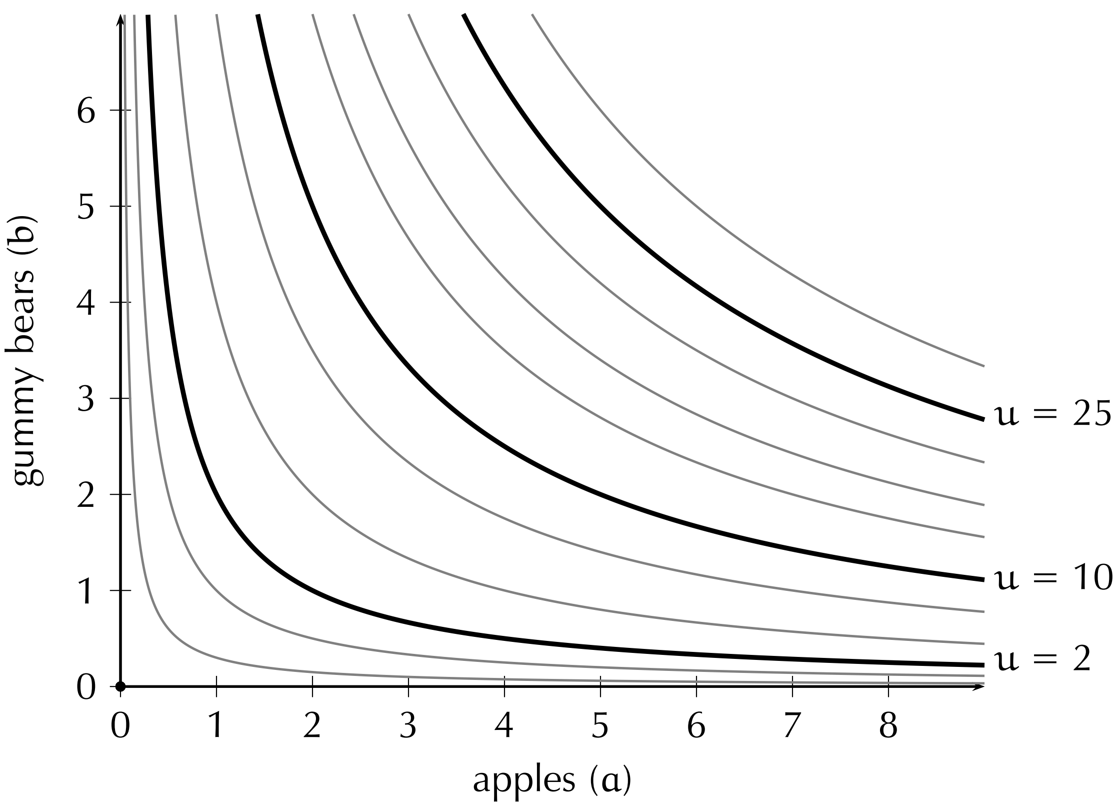
Three Cobb-Douglas Indifference Curves



4.3.2 More of the Indifference Map

The set of all indifference curves is called the **indifference map**. The figure below shows more of the indifference map for the Cobb-Douglas utility function $u(a, b) = ab$.

Partial Cobb=Douglas Indifference Map



4.3.3 Properties of Indifference Curves

The basic properties of preferences translate into conditions on indifference curves.

1. **Completeness:** There's an indifference curve through every consumption bundle.
2. **Monotonicity:** Higher indifference curves are better.
3. **Transitivity:** Indifference curves don't cross.
4. **Diminishing Subjective Value:** Indifference curves are convex toward the origin.

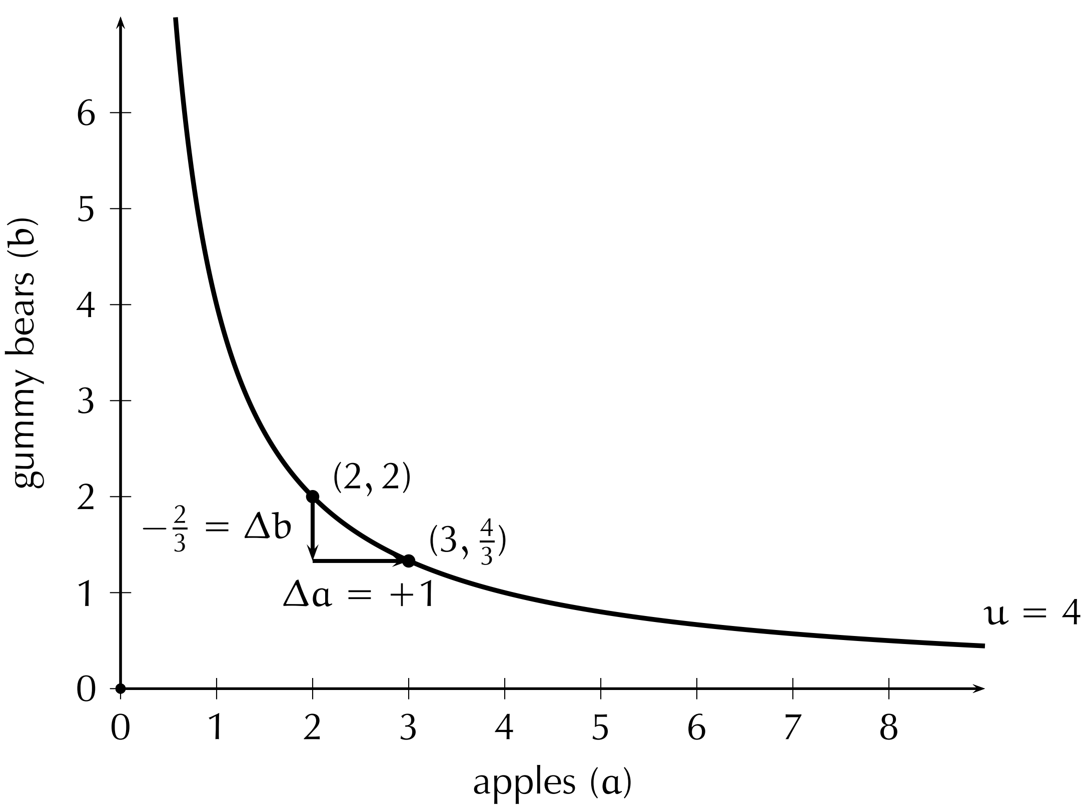
Convex toward the origin means if they are not straight, they bow out toward the origin.

4.3.4 Trade-offs and Indifference Curves

The indifference curve below ($u(a, b) = ab = 4$) describes a trade-off, showing us how many more apples it takes to compensate for the loss of a gummy bear, or in reverse, how many apples the consumer is willing to give up to get one more gummy bear.

We move from $(2, 2)$ to $(3, \frac{4}{3})$ on the indifference curve $u = 4$. Utility is unchanged, but we have gained one apple and given up two-thirds of a gummy bear. Here $\Delta a = +1$ indicates the gained apple and $\Delta b = -\frac{2}{3}$ the lost two-thirds of a gummy bear.

Trade-Off along an Indifference Curve



4.3.5 Good and Bad Trades

You can think of this trade as trading two-thirds of a gummy bear to get one apple. It says the *gummy bear price* you are willing to pay to get one apple is two-thirds of a gummy bear. In principle, it is not different from pricing goods in dollars, pesos, soles, etc.

If you give up more than two-thirds of a gummy bear, you are paying too much for the apple and end up on a lower indifference curve. A one for one trade to $(3, 1)$ would reduce your utility from 4 to 3.

On the other hand, if you give up less than two-thirds of a gummy bear for an apple, you gain. Trading half a gummy bear for one apple would bring you to $(3, 1.5)$, raising your utility to 4.5.

Try it yourself. Make up a gummy bear price of apples that is less than two-thirds apples/gummy bear. Starting from $(2, 2)$, show that utility increases if you buy an apple and see that buying gummy bears at that price increases utility.

Try selling an apple at the same price. It reduces utility. Finally, still starting at $(2, 2)$, try selling an apple at the price of one apple per gummy bear. What happens to utility.

4.3.6 Marginal Utility and Trade-offs

We can approximate the changes in utility using marginal utility. Suppose we make the trade from $(2, 2)$ to $(2 + \Delta a, 2 + \Delta b)$, gaining $\Delta a > 0$ apples at the cost of $-\Delta b > 0$ gummy bears.

Recall that marginal utility is the rate of utility change, so utility goes up by roughly $MU_a \Delta a > 0$ from having $\Delta a > 0$ more apples. The number of gummy bears is reduced by $\Delta b < 0$. Losing $-\Delta b$ gummy bears increases utility by $MU_b \Delta b < 0$. Putting this together, the change in utility is

$$MU_a \Delta a + MU_b \Delta b = 0.$$

The net change is zero because we end up back on our original indifference curve. The means that

$$MU_a \Delta a = -MU_b \Delta b,$$

so that the slope of the change is

$$\frac{\Delta b}{\Delta a} = -\frac{MU_a}{MU_b}.$$

This calculation gives us the slope of the tangent to the indifference curve at any (a, b).

4.3.7 The Marginal Rate of Substitution

The **marginal rate of substitution** (MRS) at a consumption bundle is the negative of the slope of the tangent to the indifference curve at that bundle. It is defined by

$$\mathrm{MRS} = \frac{\mathrm{MU}_a}{\mathrm{MU}_b},$$

so that

$$\frac{\Delta b}{\Delta a} = -\,\mathrm{MRS}\,.$$

In our problem, the MRS is the gummy bear **subjective marginal value** of an apple. Diminishing subjective value means that the marginal rate of substitution falls as you increase your apple consumption.

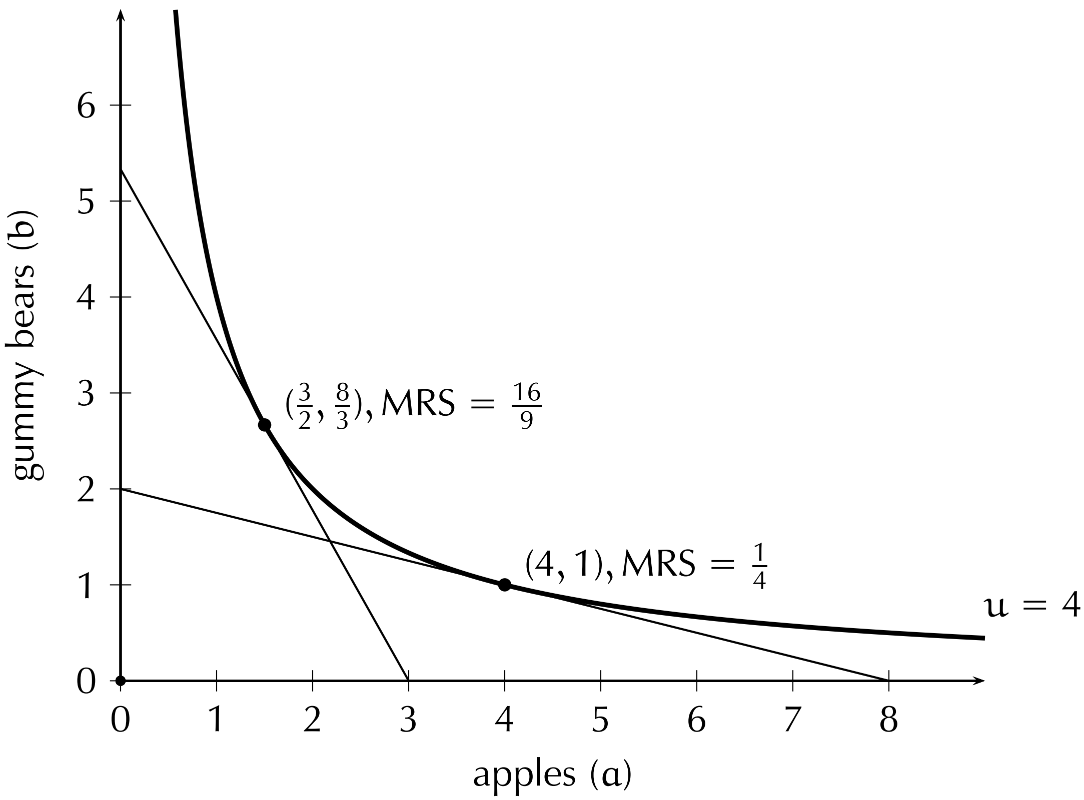
The marginal rate of substitution always measures the marginal subjective value of the good on the horizontal axis (a).[4]

[4] Sometimes, this is written MRS_{ab}, when $\mathrm{MRS}_{ab} = 1/\,\mathrm{MRS}_{ba}$. We will not otherwise use this notation.

4.3.8 Marginal Rate of Substitution as Tangent

Two such tangents are shown on the graph below, at $(\frac{3}{2}, \frac{8}{3})$ and $(4, 1)$. The former has slope $-\frac{1}{4}$ and the latter, $-\frac{16}{9}$.

Tangents and the Marginal Rate of Substitution



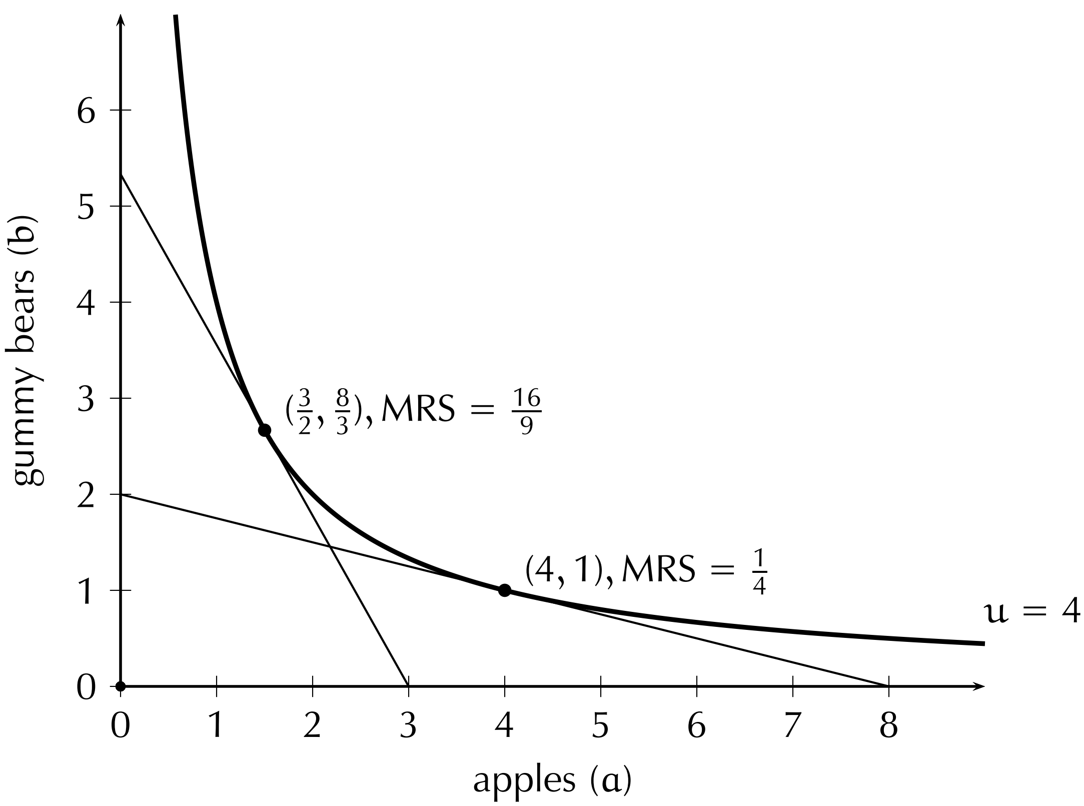
4.3.9 Computing the Marginal Rate of Substitution: Cobb-Douglas

For our Cobb-Douglas utility function $u(a, b) = ab$, we found that $MU^u_a = b$ and $MU^u_b = a$. Using the definition,

$$MRS^u = \frac{MU^u_a}{MU^u_b} = \frac{b}{a}.$$

The marginal rate of substitution $\frac{b}{a}$ is a decreasing function of a, showing that these preferences have diminishing subjective value.

Two Cobb-Douglas Marginal Rates of Substitution



NB: The marginal rate of substitution is the negative or absolute value of the slope of the tangent line.

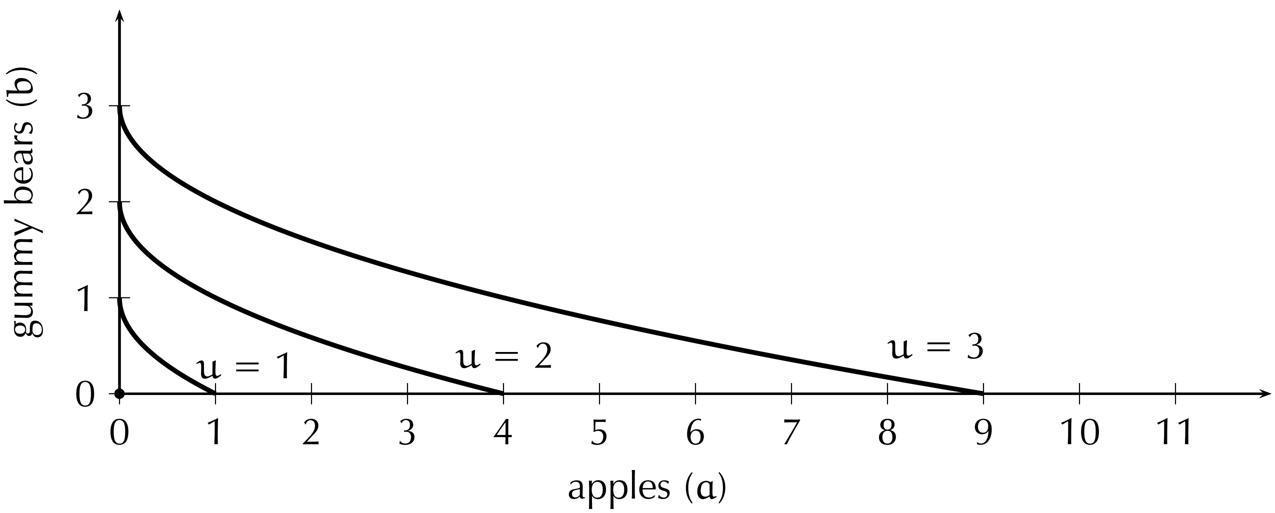
4.3.10 Computing the Marginal Rate of Substitution for v

Another utility function we looked at was $v(a, b) = b + \sqrt{a}$. We found $MU_a^v = 1/2\sqrt{a}$ and $MU_b^v = 1$. Then the marginal rate of substitution is

$$MRS^v = \frac{MU_a^v}{MU_b^v} = \frac{1}{2\sqrt{a}}.$$

Here too, $\frac{1}{2\sqrt{a}}$ is decreasing in a, meaning diminishing subjective value.

Indifference Curves for a Quasi-Linear Utility



One unusual property of these indifference curves is that slope (MRS) depends only on the quantity of apples consumed.

4.3.11 Computing the Marginal Rate of Substitution III

Another type of utility function consists of linear functions. Here's an example.

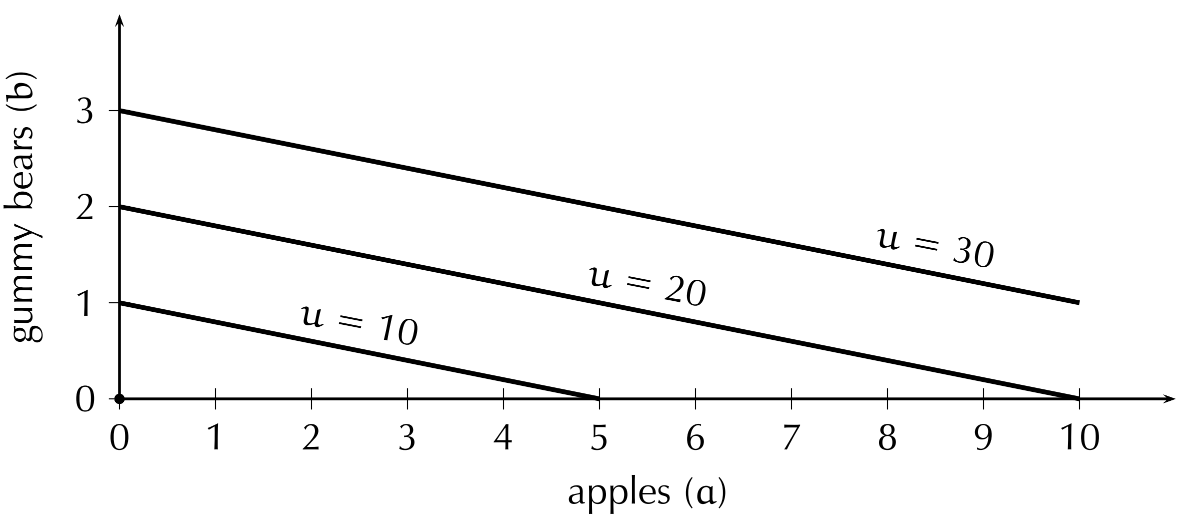
$$w(a, b) = 2a + 10b.$$

Then $MU_a = 2$ and $MU_b = 10$. So

$$MRS^w = \frac{MU_a^w}{MU_b^w} = \frac{2}{10} = \frac{1}{5}.$$

In this case the marginal rate of substitution is constant, indicating that the indifference curves are parallel straight lines. The MRS neither increases or decreases.

Indifference Curves for a Linear Utility



4.3.12 The Steepness of the Indifference Curve

The marginal rate of substitution measures the steepness of the indifference curve. A steeper indifference curve indicates that the horizontal good has higher marginal value, a flatter curve indicates lower marginal value.

If the indifference curve is horizontal, the MRS, the marginal value, is zero. Vertical indifference curves indicate an infinite marginal rate of substitution, meaning that no amount of the other good is sufficient to compensate you for giving up an apple.

As the MRS increases, its slope, or marginal subjective value, becomes higher. Indifference curves always get flatter (or at least not steeper) as we move rightward and down the indifference curve. This reflects the fact that the marginal subjective value does not increase.

4.3.13 How Curved is the Indifference Curve?

The curvature of the indifference curve indicates whether goods are substitutes or complements. Let's take utility that belongs to the Cobb-Douglas family. It can be written,

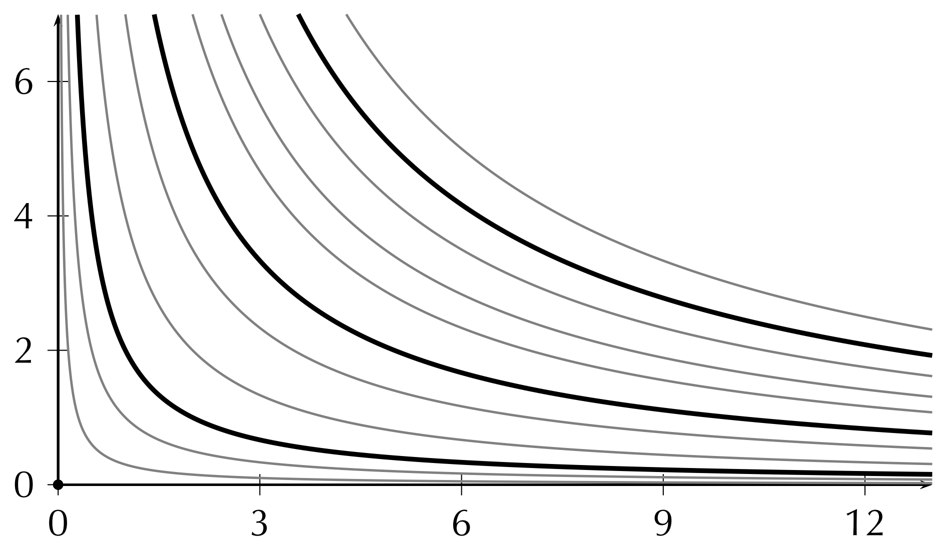
$$u_1(x, y) = Ax^a y^b$$

for some $A, a, b > 0$, the goods x and y are neither substitutes nor complements.

If the indifference curves are more sharply curved than Cobb-Douglas, the goods are complements. If they are less sharply curved, the goods are substitutes.

As a reminder, I include a Cobb-Douglas indifference map.

Partial Cobb=Douglas Indifference Map



4.3.14 Perfect Substitutes

Linear utility functions

$$u_2(x, y) = ax + by$$

with $a, b > 0$ indicate goods are perfect substitutes

Here the marginal rate of substitution

$$\text{MRS} = \frac{a}{b}$$

is constant, indicating that the utility trade-off between a and b never changes. It is fixed at a rate of $\frac{a}{b}$ units of b for one unit of a. We can always substitute $\frac{a}{b}$ units of b for a without affecting utility. We do not require the exchange rate to be 1:1 for perfect substitutes, only that it be constant.

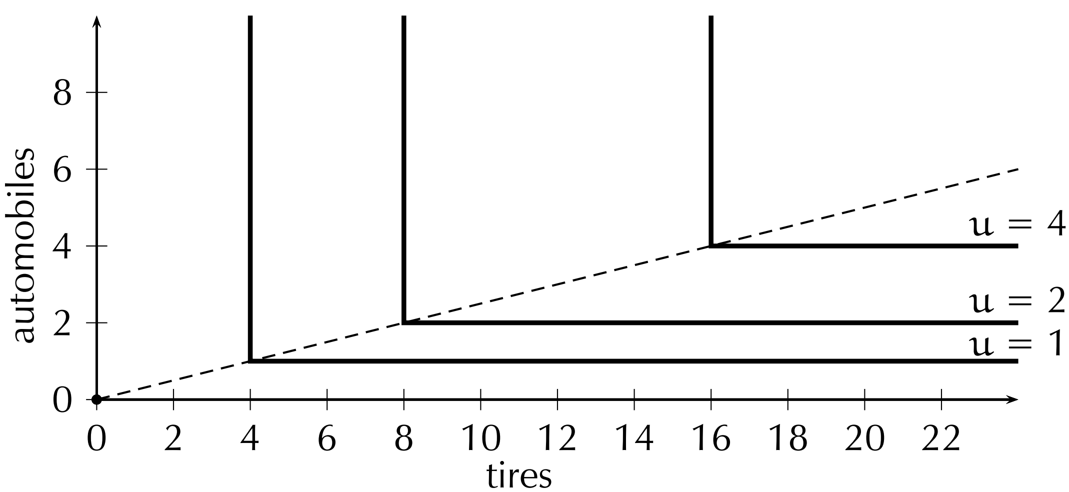
4.3.15 Perfect Complements I

Perfect complements have indifference curves that have gone beyond curving and actually fold. They are L-shaped. We start with a ratio of complementarity. In our example, it will be 4:1. Four tires are complementary to one automobile. The dashed line on the figure below shows the consumption bundles where both goods are fully utilized—where there is exactly enough of each to be used with the other, but no extra. These are the corners of the L's.

The horizontal and vertical portions of the indifference are consumption bundles where we have more than we can use of one of the goods (i.e., tires without a car to put them on, or cars without any tires to run them on).

A utility function describing the situation below is $u(x, y) = \min\{x/4, y\}$ where x is the number of tires and y the number of automobiles.

Perfect Complements

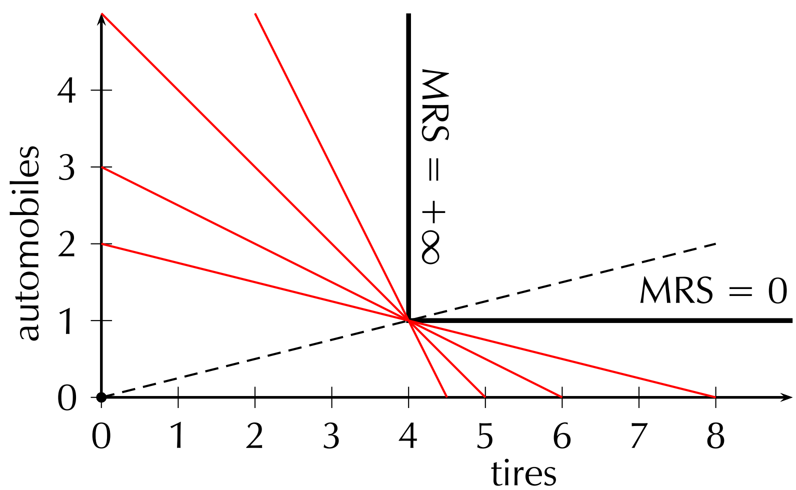


To see that these are **perfect** complements, start at $(4, 1)$. If we add tires, moving to $(5, 1)$, $(6, 1)$, $(7, 1)$, or even $(8, 1)$, utility remains at $u = 1$. To increase utility, we must add both automobiles and tires. Adding tires alone, or autos alone does nothing to increase utility.

4.3.16 Perfect Complements II

The marginal rate of substitution for perfect complements is zero on the horizontal part of the indifference curve, and is $+\infty$ on the vertical. The horizontal good contributes no extra value on the horizontal part. The vertical good contributes no extra value on the vertical part.

Multiple Tangencies with Perfect Complements



But what about the corner? The marginal rate of substitution is not defined there because there are multiple tangent lines. Several are illustrated in the diagram.

4.4 The Consumer's Budget Constraint

Having examined preferences, we now turn to the second pillar of consumer choice—the budget constraint.

The basic budget constraint is constructed from two things, prices and income. These determine how much the consumer can afford.

We continue to focus on the two good case, calling the amounts of the goods x and y. The corresponding prices are $p_x > 0$ and $p_y > 0$. The consumer also has income or wealth m available to spend.

The basic rule is that that spending cannot exceed income.

$$\text{Spending} \leq \text{Income}.$$

4.4.1 Graphing a Budget Set I

Suppose the price of X is $p_x = \$2$, the price of Y is $p_y = \$5$, and you have $m = \$10$ income to spend.

To graph a budget set, we start with two questions:

1. What is the maximum amount of y you can afford?
2. What is the maximum amount of x you can afford?

4.4.2 Graphing a Budget Set II

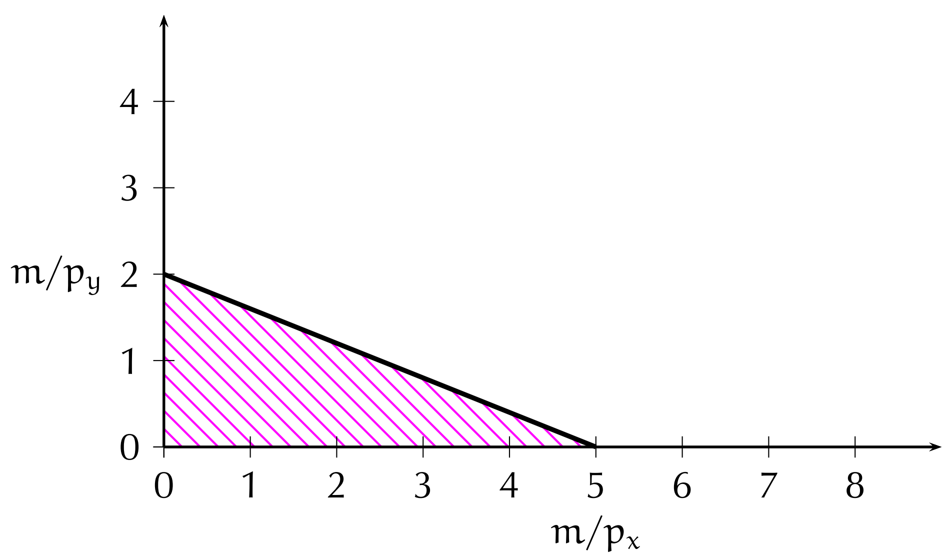
Suppose the price of X is $p_x = \$2$, the price of Y is $p_y = \$5$, and you have $m = \$10$ income to spend.

To graph a budget set, we start with two questions:

1. What is the maximum amount of y you can afford? $\mathbf{y = 2}$.
2. What is the maximum amount of x you can afford? $\mathbf{x = 5}$.

That gives us two **anchor points**, $(\mathbf{0, 2})$ and $(\mathbf{5, 0})$. These two points determine a line, the **budget frontier** or **budget line**. We fill in below it to find all the bundles we can afford, the **budget set**.

Graphing a Budget Set



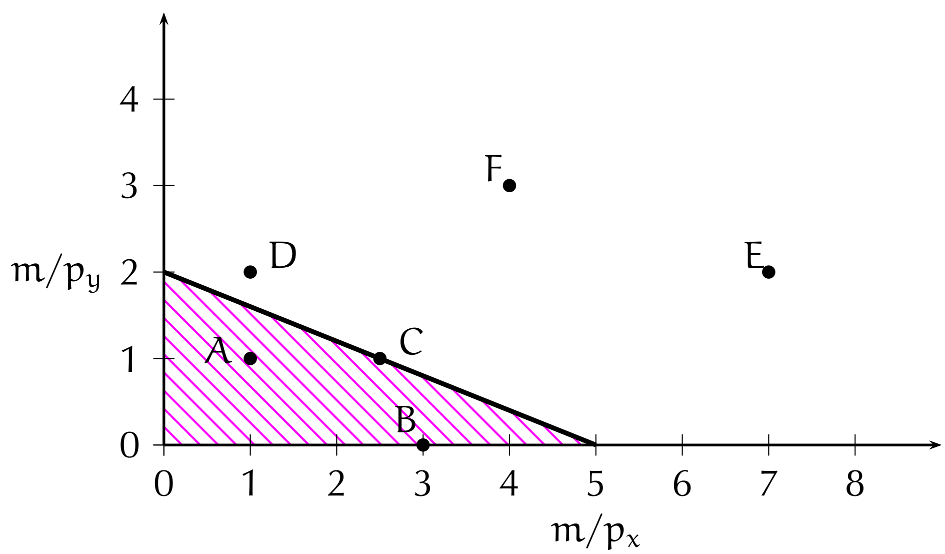
4.4.3 Limits of the Budget Set

Here $p_x = \$2$, $p_y = \$5$, and $m = \$10$.

Points such as $A = (1, 1)$ (\$7), $B = (3, 0)$ (\$6), and $C = (2.5, 1)$ (\$10) are inside the budget set. With our income, we can afford them at these prices.

Points such as $D = (1, 2)$ (\$12), $E = (7, 2)$ (\$24), and $F = (4, 3)$ (\$23) are outside the budget set. With our income, we cannot afford them at these prices.

Inside and Outside the Budget Set



Our model of the consumer is that the consumer chooses the bundle that gives the consumer the highest possible utility inside the budget set.

4.4.4 Prices, Spending, and Income

Let's approach the budget set in a more systematic fashion.

If the consumer buys x amount of good X, and faces a price of p_x, the spending on X is $p_x x$. Similarly, the spending on Y is $p_y y$. Total spending is $p_x y + p_y y$. We can now write the budget constraint as

$$p_x x + p_y y = \text{Spending} \leq \text{Income} = m$$

or more concisely,

$$p_x x + p_y y \leq m.$$

The set of all consumption bundles (x, y) that the consumer can afford at current prices and income is called the **budget set**.

4.4.5 The Budget Frontier

To graph the budget set, we start with the **budget frontier**. The set of consumption bundles that require spending all of the consumer's income. That is, those that obey the equation

$$p_x x + p_y y = m.$$

This is clearly the equation of a straight line. Let's put it into slope-intercept form by solving for y.

Then

$$p_y y = m - p_x x$$

so

$$y = \frac{m}{p_y} - \left(\frac{p_x}{p_y}\right) x.$$

4.4.6 The Relative Price

The absolute (negative) slope $\frac{p_x}{p_y}$ is called the **relative price**. It's dimensions are dollars per x-unit divided by dollars per y-unit, which reduces to y-units per x-unit, just like the marginal rate of substitution. It measures how much a unit of good X costs in terms of the good Y.

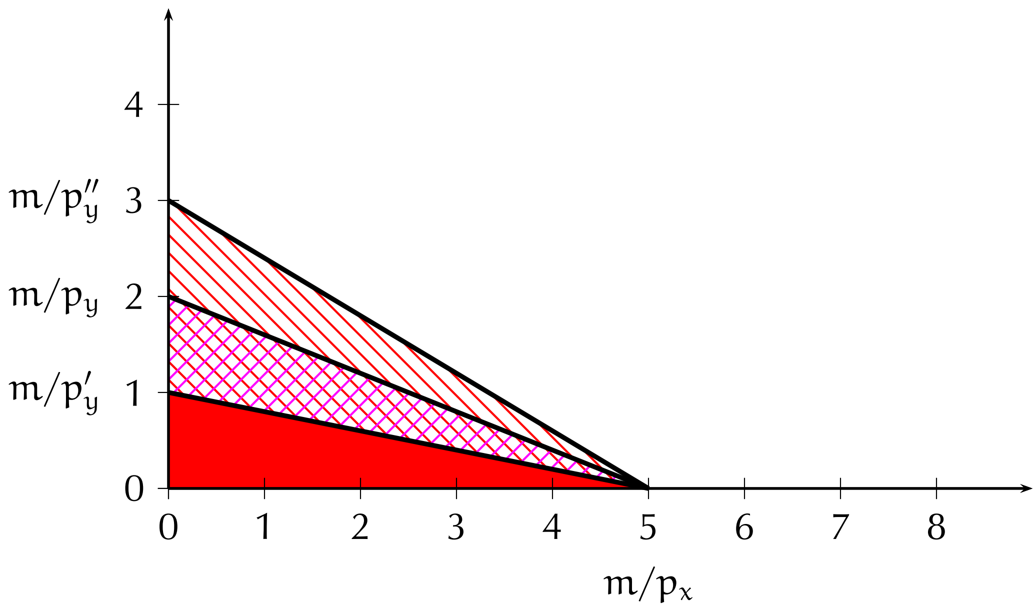
The vertical intercept is the point $(0, \frac{m}{p_y})$.

The easiest way to graph the budget line is to use its two endpoints (**anchor points**). These are the vertical intercept $\left(0, \frac{m}{p_y}\right)$ and the horizontal intercept $\left(\frac{m}{p_x}, 0\right)$.

4.4.7 Changes in p_y

We consider the budget frontier under three different prices: $p'_y = \$10$, $p_y = \$5$, and $p''_y = \$3.33$.

Changing Budget Set with p_y



The price of X has not changed, so the X anchor-point remains fixed. The budget frontier rotates about that anchor point. A higher price of p_y rotates the budget line downward, while a lower price rotates it upward.

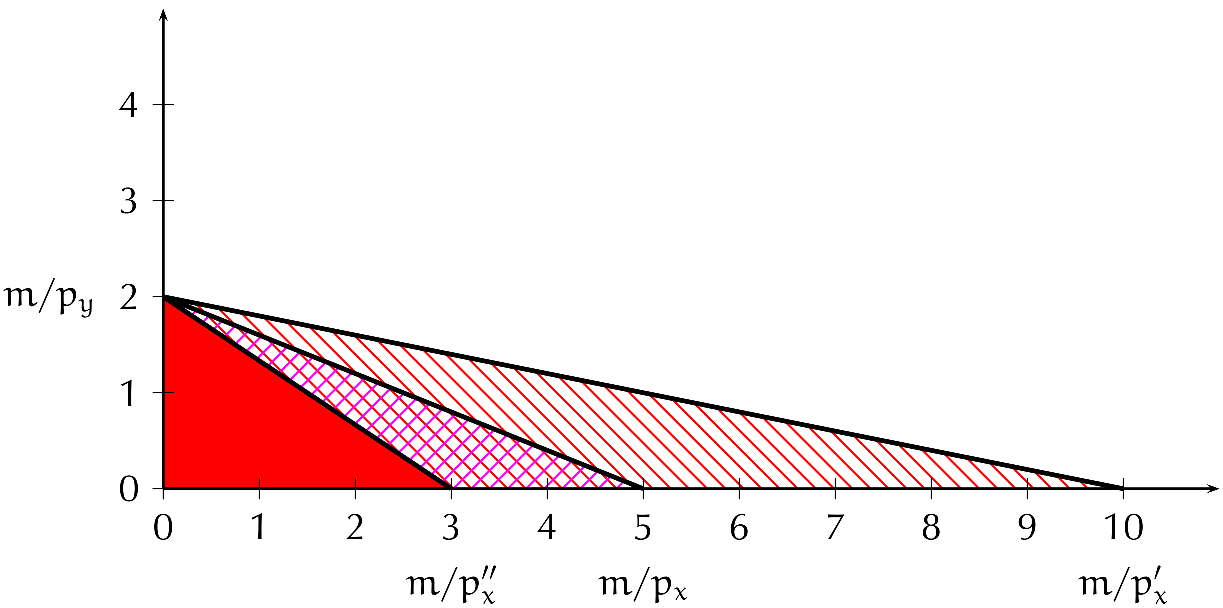
The shaded areas are the budget sets. The budget set gets smaller as the price of Y increases, larger as the price decreases. Since the options available are increased at lower prices, we can say that real income (what you can buy with your income) has gone up.

4.4.8 Changes in p_x

We consider the budget frontier under three different prices: $p'_x = \$1$, $p_x = \$2$, and $p''_x = \$3.33$.

The price of Y has not changed, so the Y anchor-point remains fixed. The budget frontier rotates about that anchor point. A higher price of p_x rotates the budget line left, while a lower price rotates it right.

Changing Budget Set with p_x



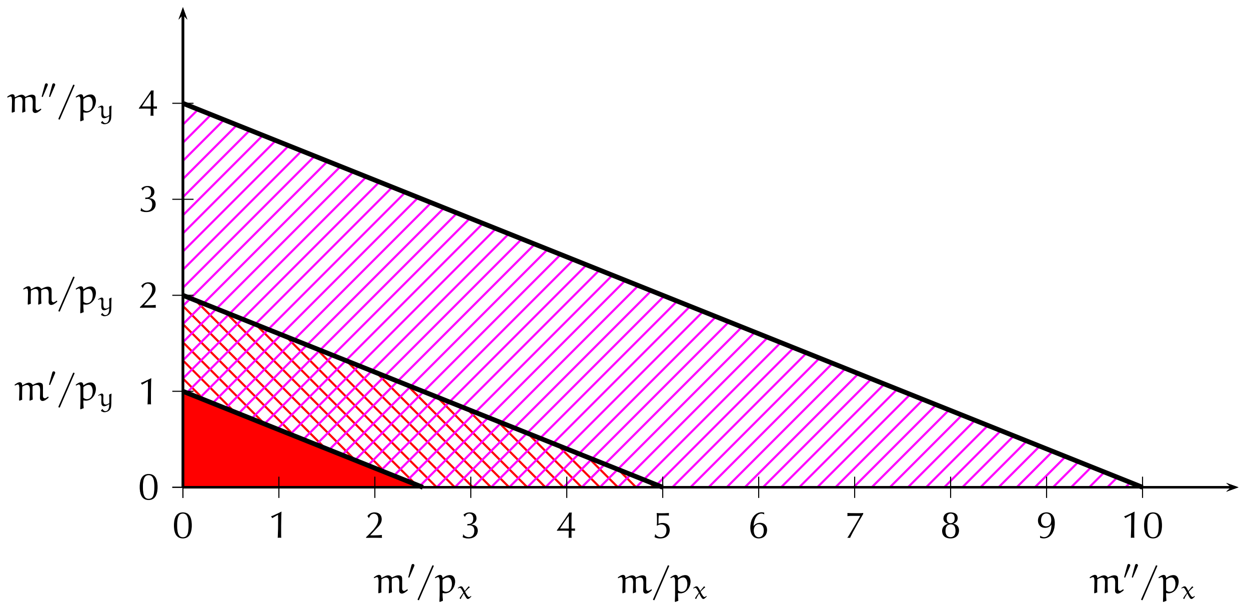
The shaded areas are the budget sets. The budget set gets smaller as the price of X increases, larger as the price decreases. Since the options available are increased at lower prices, we can say that real income (what you can buy with your income) has gone up.

4.4.9 Changes in Income

We consider the budget frontier under three different incomes: $m' = \$5$, $m = \$10$, and $m'' = \$20$.

Changes in income lead to parallel shifts in the budget frontier.

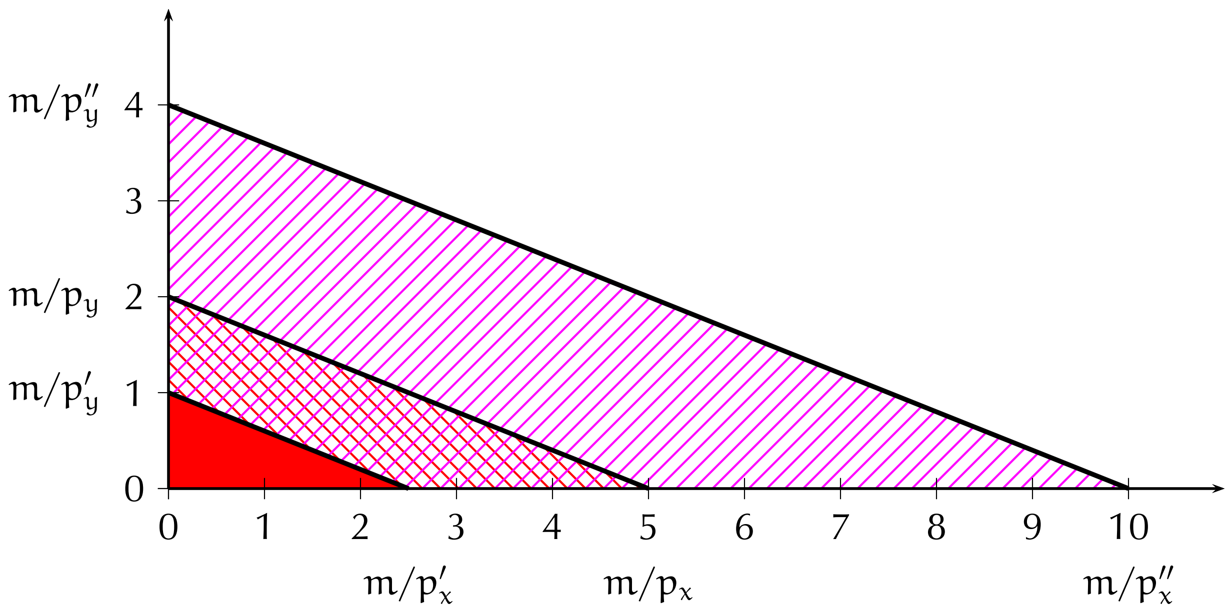
Changing Budget Set with income



4.4.10 Changes in Both Prices

Finally, we look at the case where both prices increase in the same proportion. Besides $(p_x, p_y) = (2, 5)$, we examine $(p'_x, p'_y) = (4, 10)$ (both doubled) and $(p''_x, p''_y) = (1, 2.5)$. The result is again a parallel shift in the budget frontier that is identical to our income changes.

Changing Budget Set with proportional Price Changes



In fact, if $(p'_x, p'_y) = a(p_x, p_y)$, the anchor points with income m are $\frac{1}{a}(m/p_x)$ and $\frac{1}{a}(m/p_y)$, which are identical to the anchor points with income m/a and the original prices (p_x, p_y). Proportional price changes and income changes have the same effect on prices.

September 16, 2022