Intermediate Microeconomics — Week 7

Professor Boyd **Oct. 4 & 6, 2022**

QUIZ: Thursday's quiz covers weeks 5 and 6, including sections 4.4 through the end of Chapter 5 (pp. 117–179).

5.5 A Model of Borrowing and Lending **Repeated**

Another application of consumer theory is to borrowing and lending. In the latter case, the theory addresses consumers who are suppliers to the loan market—suppliers of loans. In other words, they are savers. It also is concerned with consumers who are demanders of loans (borrowers).

5.5.1 A Two-Period Model

We will use a simple model with two time periods, the present and the future, also known as today and tomorrow.

We distinguish consumption in each period. Consumption now is c_0 and future consumption is c_1. Both are assumed to be normal goods.

The consumer earns income in each period, w_0 now and w_1 in the future.

Finally, there is an interest rate $r > 0$. All borrowing and lending occurs at the interest rate r.

5.5.2 The Budget for Borrowers I

Repeated

Suppose our consumer decides to borrow money from the bank today. Let A be the amount borrowed. Borrowing an amount A today (the principal) commits the consumer to paying the lender back in the future. The amount paid back consists of both the principal A plus interest $r \times A$. That means the borrower must pay back

$$A + rA = (1 + r)A$$

in period one.

Borrowing in period one isn't possible because there is no period two when it can be paid back.[1]

Having borrowed A, consumption in period zero can be increased from w_0 to $c_0 = w_0 + A$. That means that the consumer borrows

$$A = c_0 - w_0 > 0$$

in period zero and pays back

$$(1 + r)A = (1 + r)(c_0 - w_0)$$

in period one.

[1] More complex versions of this model allow for borrowing in many periods.

5.5.3 The Budget for Borrowers II

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There's no borrowing in the future (period one), so the consumer spends their remaining income on consumption. Thus

$$c_1 = w_1 - (1 + r)(c_0 - w_0). \tag{5.5.1}$$

We assume that no one will lend to anyone with insufficient income to pay back the loan. In other words, we require that $w_1 \geq (1+r)(c_0 - w_0)$. This can be rewritten

$$c_1 = w_1 - (1 + r)(c_0 - w_0) \geq 0.$$

We can now rewrite equation (5.5.1) as follows

$$c_1 + (1 + r)c_0 = w_1 + (1 + r)w_0$$

or if we divide by $(1 + r)$, we obtain

$$c_0 + \frac{c_1}{1 + r} = w_0 + \frac{w_1}{1 + r}. \tag{5.5.2}$$

Dividing by $(1 + r)$ is called **discounting**. When we add the current value of goods now and the discounted value of goods in period one we obtain the **present value** of those goods. Equation (5.5.2) says that the present value of consumption (c_0, c_1) is equal to the present value of income (w_0, w_1).

5.5.4 The Budget for Savers (Lenders) I

If you save an amount A at a bank, you are lending them the amount A in period zero. In return, the bank will pay you back the principal A together with the interest $r \times A$, for a total repayment of $(1 + r)A$.

Saving A in period zero means the consumer has only $w_0 - A$ available for consumption c_0, so $c_0 = w_0 - A$, or equivalently, $A = w_0 - c_0 > 0$.

In the future, the consumer gets their principal $(w_0 - c_0)$ back, plus interest $r(w_0 - c_0)$. This increases future income to

$$w_1 + (1 + r)(w_0 - c_0).$$

The income is all spent on future consumption, so

$$c_1 = w_1 + (1 + r)(w_0 - c_0) \tag{5.5.3}$$

5.5.5 The Budget for Savers (Lenders) II

Rearranging equation (5.5.3), we find

$$(1 + r)c_0 + c_1 = (1 + r)w_0 + w_1$$

or, dividing by $(1 + r)$, we again find

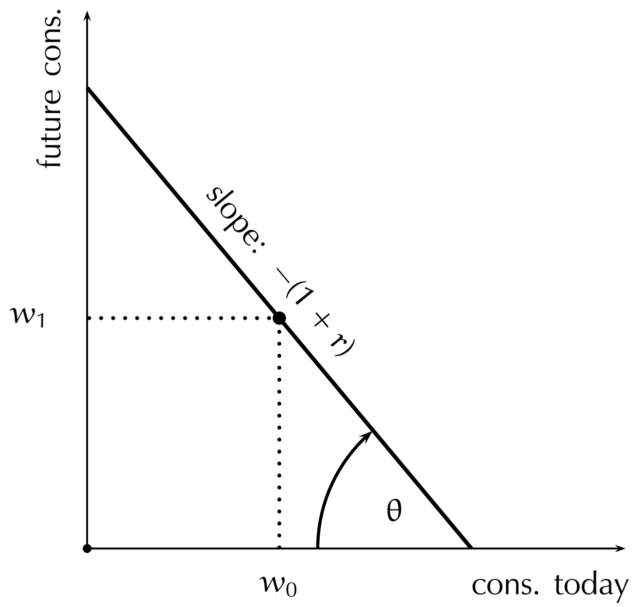
$$c_0 + \frac{c_1}{1 + r} = w_0 + \frac{w_1}{1 + r}.$$

This is the same as equation (5.5.2), showing that both **borrowers and savers face the same budget constraint**.

5.5.6 The Budget Frontier for Borrowing and Lending

The key thing about the budget frontier is that you always have the option of neither borrowing nor lending. In that case, you consume your income in each period, (w_0, w_1). Moreover, this point is on the budget frontier. It is the only anchor point. The slope of the budget frontier is $-(1 + r)$. Since $r > 0$, its angle $\theta > 45°$. This is illustrated below.

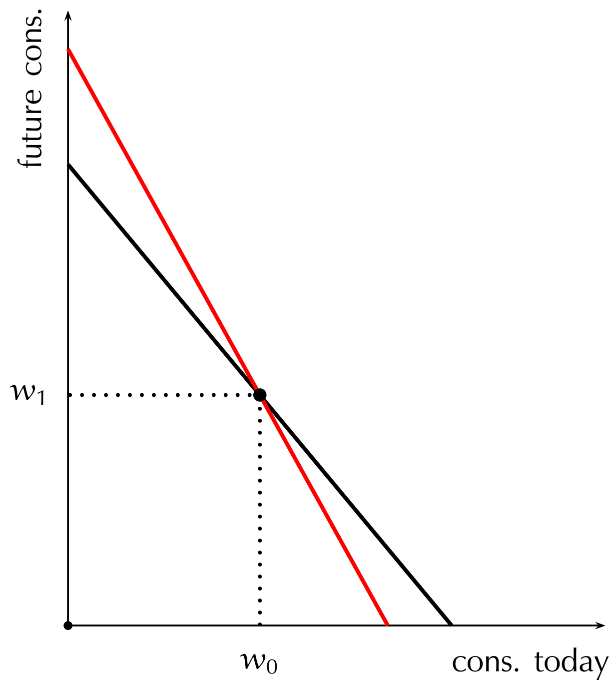
Budget Frontier with Borrowing and Lending



5.5.7 An Increase in the Interest Rate

When the interest rate increases, the budget line rotates clockwise about the anchor point (w_0, w_1). A decrease in the interest rate would rotate the budget frontier counter-clockwise.

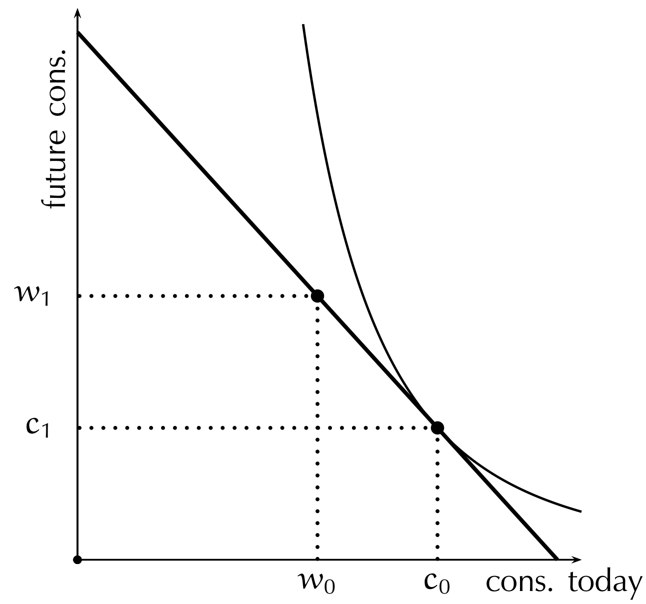
Effect of an Interest Rate Increase



5.5.8 Borrowing I

This consumer's preference map leads to borrowing at the current interest rates. To see that, compare w_0 and c_0. We see that $c_0 > w_0$, indicating that the consumer is spending more than income. That requires borrowing. Moreover, $c_1 < w_1$, so debt is being paid off in the future.

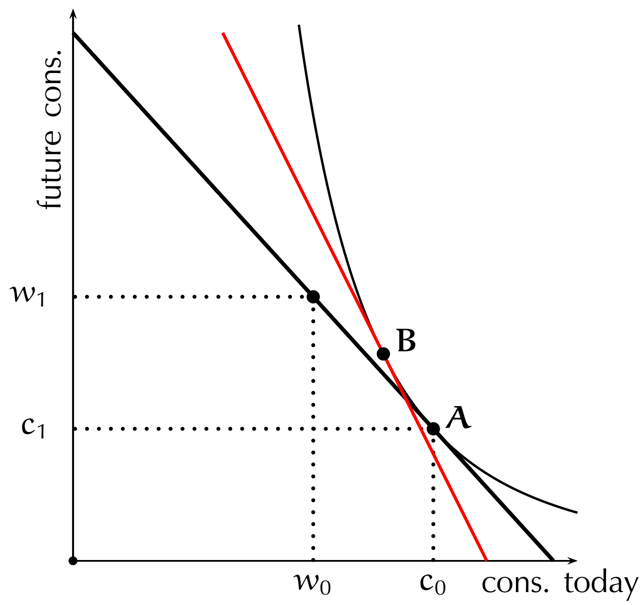
Optimal Choice with Borrowing and Lending



5.5.9 Borrowing II

The red line illustrates the substitution effect of a increase in the interest rate. This moves the consumer from point A to point B. Consumption today is decreased because the cost of borrowing has gone up.

Substitution Effect of Rate Increase

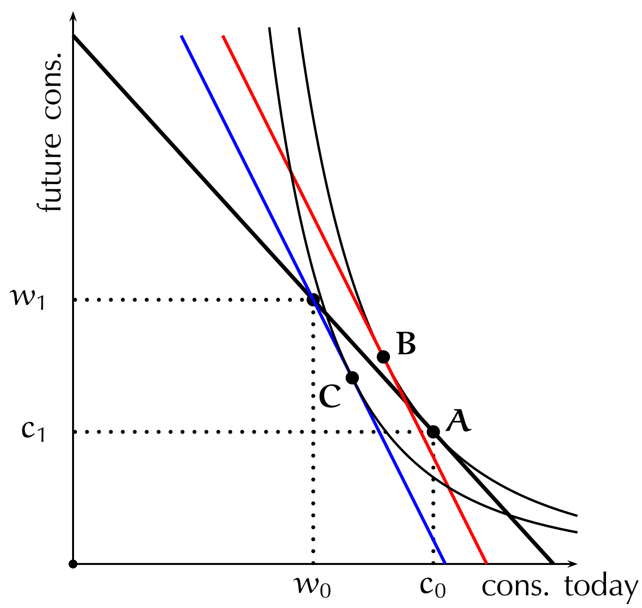


5.5.10 Borrowing III

The blue line illustrates the income effect of a increase in the interest rate. This line goes always goes through the anchor point at (w_0, w_1), and of course is parallel to the red line.

The income effect moves consumption from bundle **B** to bundle **C**. Consumption today decreases because our borrower has become poorer, and so chooses to consume less in both periods (rembember, both goods are normal). Since the substitution effect also reduces consumption today, consumption today is reduced overall, and so is borrowing. The demand curve for loans slopes downward.

Substitution and Income Effects of Rate Increase

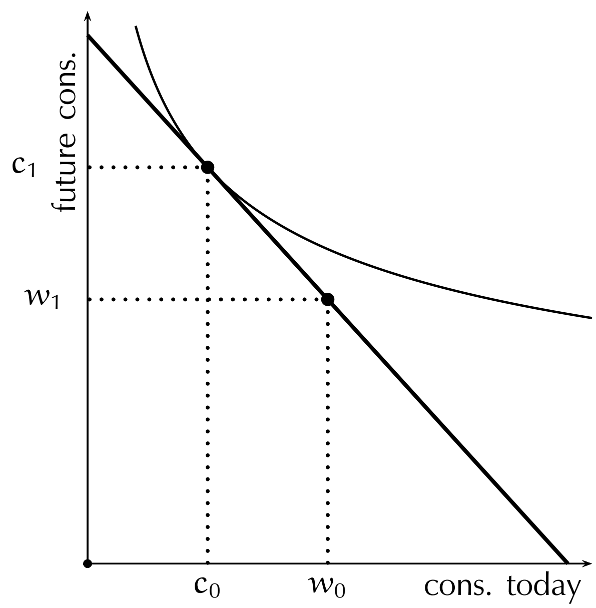


5.5.11 Saving

Suppose the interest rate increases. How does it affect savers? If we have a saver, the consumption bundle lies to the left of the income point (w_0, w_1). The substitution effect moves the consumer up to the left on their indifference curve, reducing consumption today and so increasing saving. The income effect makes the consumer richer (they earn more interest on their savings), leading to increased consumption of both goods. This reduces savings.

The overall effect on savings is ambiguous, although this consumer will end up consuming more in the future, both from the income and substitution effects. We cannot say whether the supply of savings is upward sloping. Data indicates it is actually fairly inelastic.

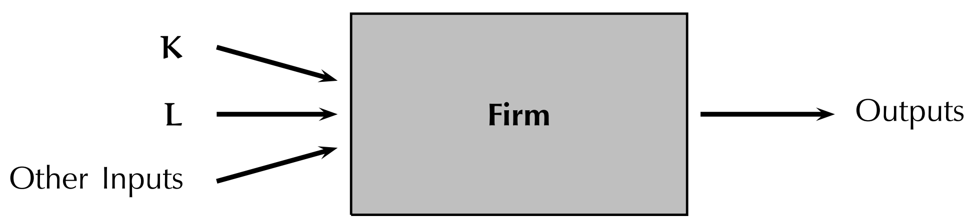
A Saver's Optimum



6.1 Chapter 6: Producer Behavior

It time to turn to supply. We've already considered some the issues that come up with labor supply and in credit markets. Here we will focus on firms that produce products for sale, either to consumers or to other producers (intermediate goods).

Firm Inputs and Outputs



We often treat the firm as a black box. Inputs (factors of production) go into the firm, and outputs come out. Of course, this being economics, all transactions are two-way streets. When goods and services come in to the firm, money goes out to the suppliers, and when goods and services go out, money comes in from their customers. In this course, we generally ignore issues involving corporate finance (stocks, bonds, etc.).

6.1.1 Simplifying Assumptions

There are nine basic assumptions that underlie our model of the firm.

1. Each firm produces a single good.
2. Each firm has already chosen that good.
3. Each firm produces their good in the cheapest way possible (cost minimization).
4. Each firm uses two inputs: we call them K and L (capital and labor).
5. In the short run, a firm can adjust L but not K. In the long run both can be changed
6. More input leads to more output.
7. The marginal product of K is non-increasing in input of K. The marginal product of L is non-increasing in input L.
8. The firm is a price taker in the input markets. The firm can buy as much or little as they wish at the current market price.
9. The is no budget constraint for the firm. This amounts to assuming they can borrow to finance current operations.

6.1.2 Production Functions

A production function tells us how much a firm can produce from given quantities of inputs. We write the quantity of output produced as

$$Q = F(L, K)$$

Examples of simple production functions with two inputs include:

$Q = F(L, K) = \sqrt{KL}$	Basic Cobb-Douglas
$Q = F(L, K) = AK^a L^b$	General Cobb-Douglas, $A, a, b > 0$
$Q = F(L, K) = aK + bL$	Linear, with a, $b > 0$
$Q = F(L, K) = \min\{aK, bL\}$	Leontief, with a, $b > 0$

6.1.3 Short-run Production

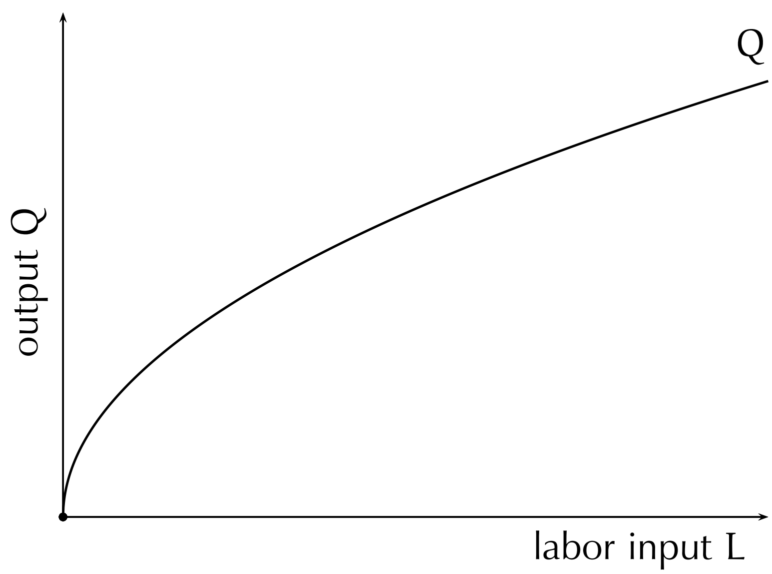
The key thing about short-run (SR) production is that **capital is fixed in the short run**. It cannot be changed (it can always be changed in the long run).

Let's fix the capital stock at the level $\bar{K}$. Then what happens to production? In the Cobb-Douglas case, we have $Q = \sqrt{KL}$.

Let's take $\bar{K} = 100$, the resulting short-run production function is

$$F(L, \bar{K}) = F(L, 100) = \sqrt{100L} = 10\sqrt{L}.$$

Short-run Production Function



6.1.4 Marginal Product of Labor

The slope of the short-run production function is the **marginal product of labor**, MP_L. We can calculate it using either finite differences or calculus.

$$MP_L = \frac{\Delta Q}{\Delta L} \quad \text{or} \quad MP_L = \frac{\partial Q}{\partial L} = \frac{dQ}{dL}.$$

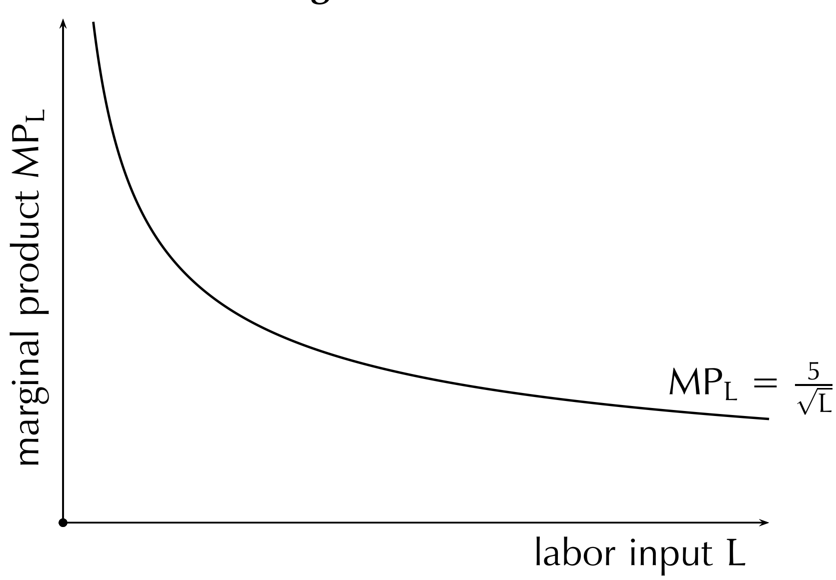
We can use the ordinary derivative here because the capital stock is a constant. The two methods shown may slightly different numbers.

In the case of the production function $10\sqrt{L}$,

$$MP_L = 10\left(\frac{1}{2}\right)L^{-1/2} = 5L^{-1/2} = \frac{5}{\sqrt{L}}$$

Notice that the marginal product for Cobb-Douglas production is decreasing in L (point 7).

Marginal Product of Labor



6.1.5 Marginal Product: Finite Differences

ΔL	L	Q	ΔQ	MP_L
–	1	10	–	–
1	–	–	4.1	4.1
–	2	14.1	–	–
1	–	–	3.2	3.2
–	3	17.3	–	–
1	–	–	2.7	2.7
–	4	20	–	–

Because $\Delta L = 1$, ΔQ and $MP_L = \Delta Q/\Delta L$ are identical in the table. That is not always the case.

6.1.6 Average Product of Labor

The **average product of labor**, AP_L, is the ratio of output to labor input. In the short-run,

$$AP_L = \frac{Q}{L} = \frac{F(L, \bar{K})}{L}$$

where K is fixed at $\bar{K}$.

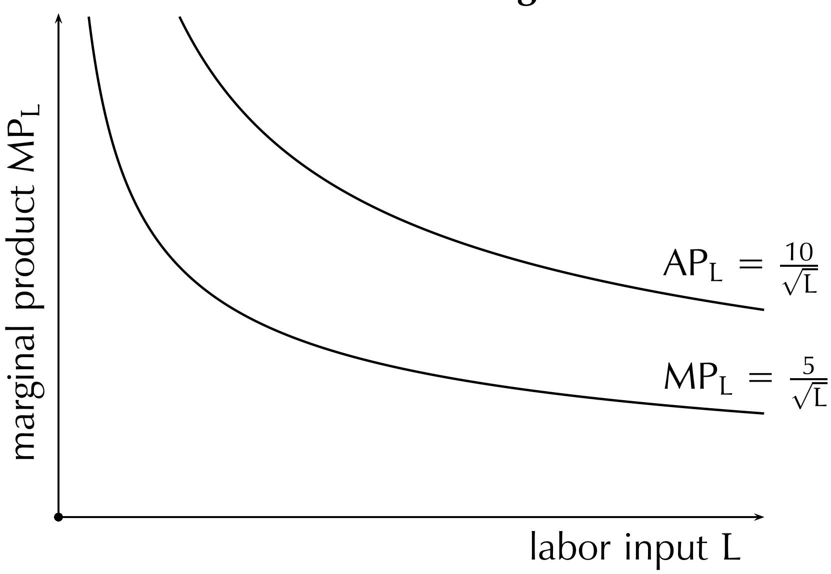
For our Cobb-Douglas production function with $\bar{K} = 100$, we have

$$Q(L) = \frac{10\sqrt{L}}{L} = \frac{10}{\sqrt{L}}.$$

6.1.7 Average and Marginal Products

Both the average and marginal products are measured using the same units, units of output per hour of labor. Since the units are the same, we can meaningfully graph them using the same vertical axis. I've done that below for our Cobb-Douglas function $F(L, \bar{K}) = 10\sqrt{L}$ where $K = 100$.

Average and Marginal Products of Labor
Cobb-Douglas



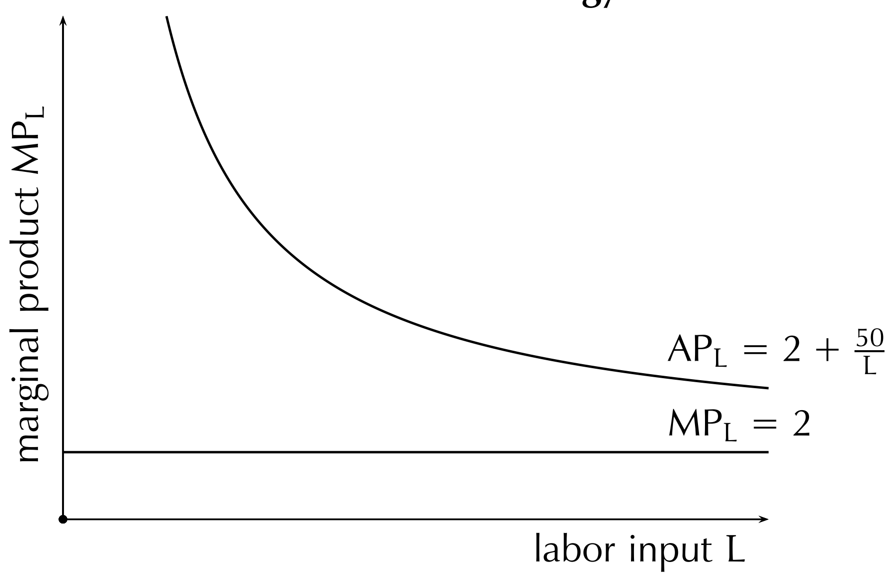
6.1.8 Average and Marginal Products with Linear Production

Let's consider the linear production function $F(L, K) = 2L + K$. We set capital to $\bar{K} = 50$ in the short run. Then $Q(L) = 50 + 2L$. The marginal and average products are:

$$MP_L = 2 \quad \text{or} \quad AP_L = 50/L + 2.$$

This is graphed below.

Average and Marginal Products of Labor
Linear Technology



6.1.9 Relation between Marginal and Average: Finite Changes

You'll notice that in both graphs, the marginal product was below the average product, and the average product was falling (has a negative slope). This is no coincidence.

When comparing marginal and average anything, the average falls when the marginal is below the average and rises when the marginal is above the average.

This is easy to see with finite changes. Consider everyone currently in this room. Compute the average height. Now suppose another person walks in with height above average. Then the average will rise. Alternatively, if that new person had height below average, the average would fall.

Suppose there were 7 people in the room with heights 62, 65, 68, 70, 71, 72, and 75 inches, the average is their sum 483 inches divided by the number of people (7). The average is $483/7 = 69$ inches $= 5'9''$. If someone with height $5'7'' = 67''$ walks in, we have 8 people with a total height of 550 inches. The average is $550/8 = 68.75 = 5'\ 8.75''$. The average has fallen because we added a person shorter than average.

6.1.10 Relation between Marginal and Average: Calculus

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The average product of labor is

$$\frac{F(L, \bar{K})}{L}.$$

We take the derivative with respect to labor.

$$\begin{aligned}
\frac{d}{dL}\left(\frac{F(L, \bar{K})}{L}\right) &= \frac{L\frac{\partial F}{\partial L} - F}{L^2} \\
&= \frac{L\frac{\partial F}{\partial L}}{L^2} - \frac{F}{L^2} \\
&= \frac{1}{L}\left(\frac{\partial F}{\partial L} - \frac{F}{L}\right) \\
&= \frac{1}{L}\left(MP_L - AP_L\right)
\end{aligned}$$

If $MP_L > AP_L$, the average product is rising. If $MP_L < AP_L$, the average product is falling. If $MP_L = AP_L$, there is no change in the average product.

6.1.11 A Second Take on Average Products

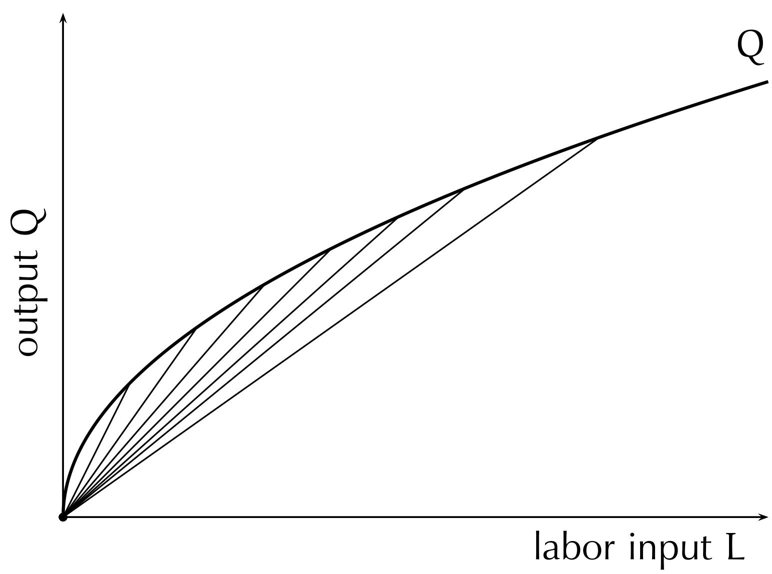
The average product of labor is defined as

$$\frac{Q(L)}{L} = \frac{Q(L) - 0}{L - 0}.$$

The point of writing this in a second way is that we can think of the average product as the slope of the chord connecting the origin $(0, 0)$ and the point $\big(Q(L), L\big)$ on the graph of $Q(L)$. This is illustrated below.

Here the average product again declines as we move to the right.

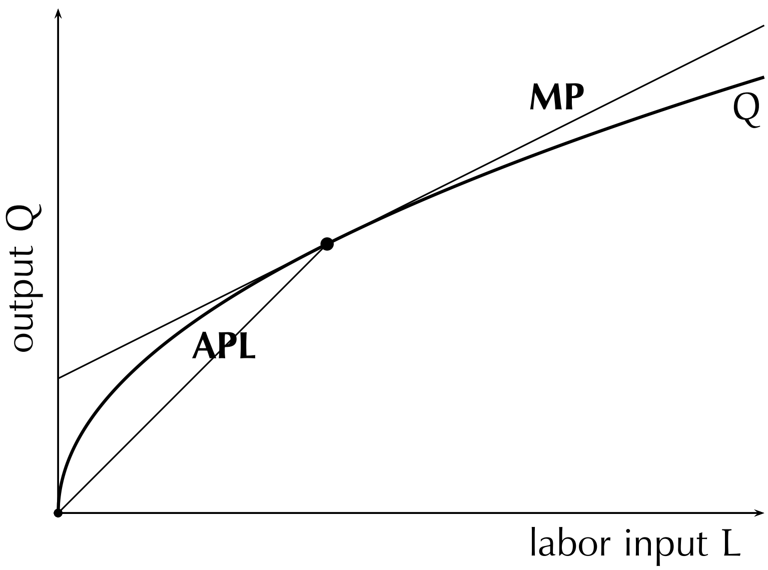
Average Product of Labor



6.1.12 Average and Marginal Product Revisited

The marginal product is the slope of the tangent, the average product is the slope of the chord from the origin. It is clear on the diagram that the the tangent's slope is smaller than the slope of the chord to zero. The picture is similar at other points on the production function, so $MP_L < AP_L$.

Average and Marginal Products of Labor



6.2 Long-run Production

In the long run, all factors are variable. In our two input model, that means both capital and labor can be varied in the long run. As with utility, properly graphing long-run production functions would require using three dimensions.

With utility, we sidestepped this by using indifference curves—the set of bundles (x, y) obeying $u(x, y) = \bar{u}$ for some number $\bar{u}$. We do a similar thing with production, focusing on **isoquants**—the combinations of inputs (L, K) that produce the same level of output. That is, for a given quantity q, we find all pairs (L, K) with $F(L, K) = q$.

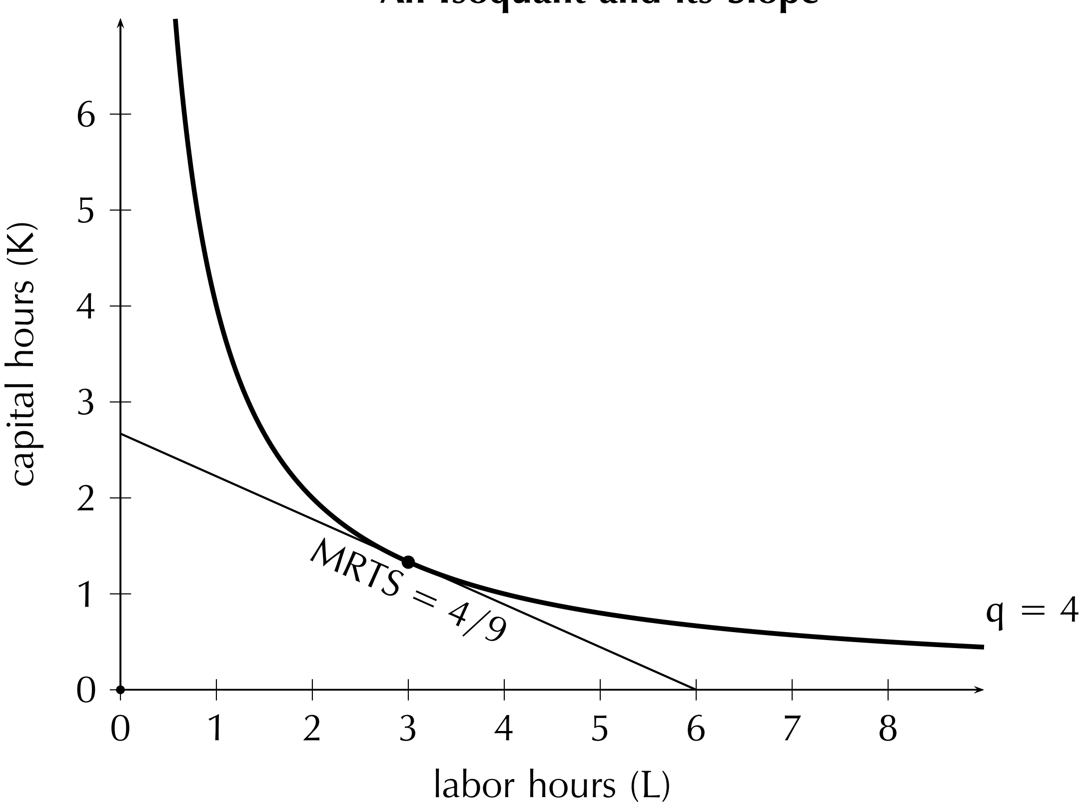
6.2.1 Isoquants

We will measure both capital and labor in terms of hours used. If a capital good is left idle, it's not considered an input.

As with indifference curves, we are interested in the slope at various points along the isoquant. For indifference curves, the absolute slope is called the marginal rate of substitution. For isoquants, the absolute slope is the **marginal rate of technical substitution**, MRTS.

We always compute the slope with L on the horizontal axis and capital on the vertical axis.

An Isoquant and its Slope



6.2.2 Computing the Marginal Rate of Technical Substitution

With indifference curves, $u(x,y) = \bar{u}$, the absolute slope of the tangent line is

$$\text{MRS} = \frac{\text{MU}_x}{\text{MU}_y}.$$

The formula is basically the same for isoquants. The isoquant obeys $F(L,K) = q$, and we simply replace the marginal utility by marginal product to find the marginal rate of technical substitution.

$$\text{MRTS} = \frac{\text{MP}_L}{\text{MP}_K}.$$

We use a different term for the absolute slope of isoquants to remind us we are thinking about producers, not consumers.

6.2.3 Cost of Inputs

By assumption (8), the firm is a price taker in the input markets. The firm can buy as much or little as they want at the current price in the input markets. There is a price w for labor, the hourly wage rate, and a price r for capital, the hourly rental rate for capital.

When the firm uses L hours of labor and K hours of capital, their cost is

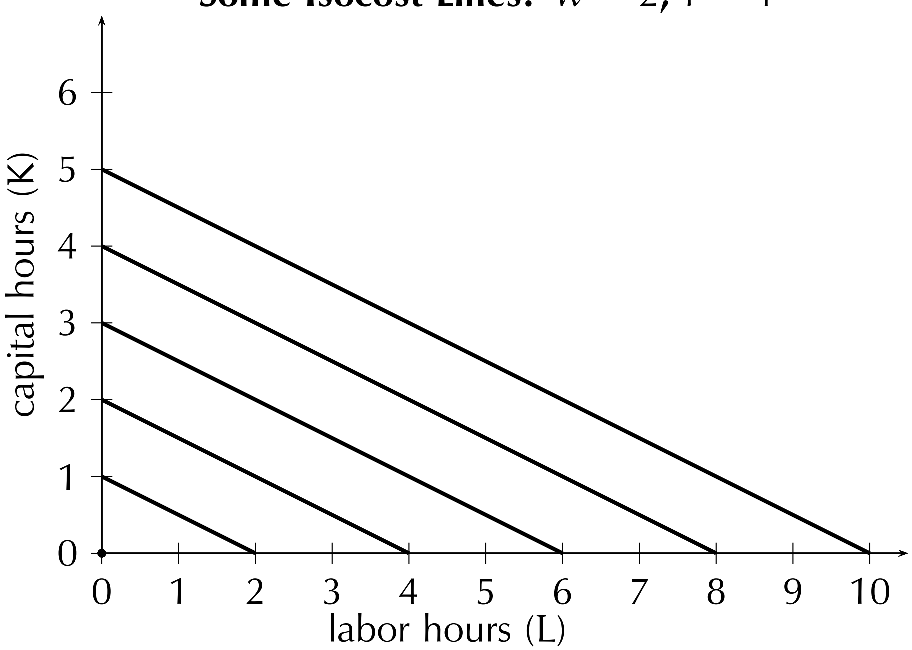
$$rK + wL.$$

An **isocost line** consists of bundles of inputs that all cost the same. That is they obey

$$rK + wL = \text{some constant}$$

The absolute slope of any of the isocost lines is given by the relative price w/r.

Some Isocost Lines: $w = 2$, $r = 4$



6.3 Long-run Production: Cost Minimization

We will eventually focus on profit maximization. A key step toward solving that problem is to find the cost function, which describes how much it costs a firm to produce a given quantity q.

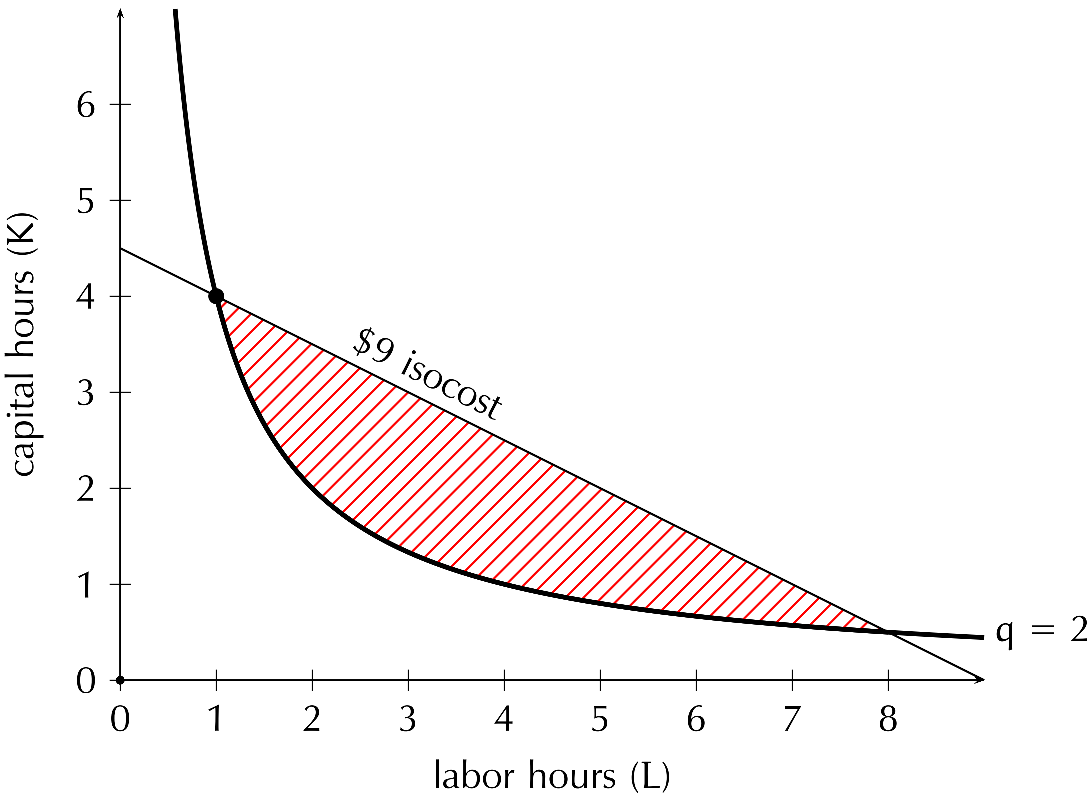
We find the cost function by using the production function F together with input prices r and w to find the least expensive way to produce any quantity q.

The cost of using inputs (L, K) is then

$$\text{Cost} = rK + wL$$

6.3.1 An Attempt to Minimize Cost

Let's try a random point on the $q = 2$ isoquant. We'll see how much it costs and see if we can do better. Suppose $r = 2$, $w = 1$ and try the input combination $(L, K) = (1, 4)$. The point $(1, 4)$ costs $w + 4r = 1 + 8 = \$9$, so we draw the \$9 isocost. The red-hatched area above the $q = 2$ isoquant and below the \$9 isocost represents input combinations that produce at least 4 units of output at less cost than $(1, 4)$.



6.3.2 Minimizing Cost

It's obvious from our previous attempt at minimizing cost that the problem is that the isocost cut through the isoquant. If they are not lined up, we will not be at a minimum. They must be tangent. Now the relative price is $w/r = 1/2$ and the slope of the isoquant is MRTS $= K/L$. It follows that

$$\frac{w}{r} = \text{MRTS} = \frac{K}{L} \quad \text{so} \quad wL = rK$$

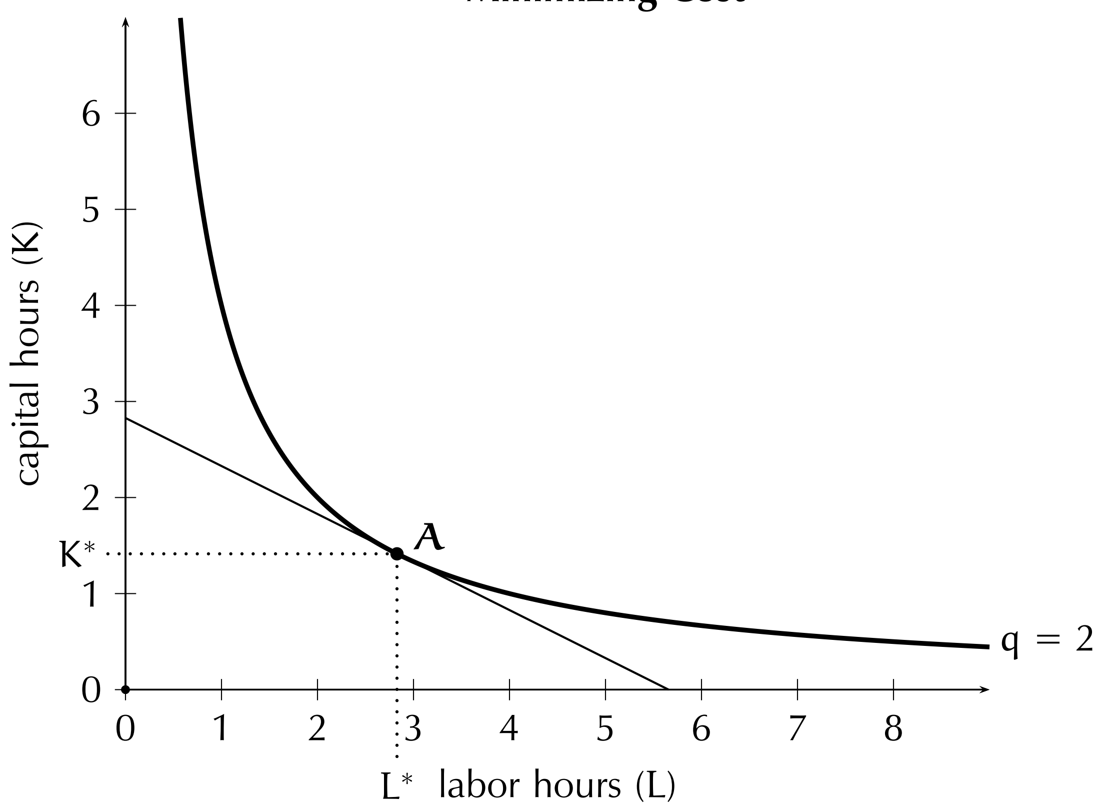
when cost is at a minimum. In this case, $2K = L$. Further, we must be on the isoquant for $q = 2$. This happens at $\mathbf{A} = (L^*, K^*)$ where

$$q = 2 = F(L^*, K^*) = F(2K^*, K^*) = \sqrt{2(K^*)^2} = K^*\sqrt{2}$$

so $(L^*, K^*) = (2\sqrt{2}, \sqrt{2})$ with cost

$$C(2) = w(2\sqrt{2}) + r\sqrt{2} = 4\sqrt{2}.$$

Minimizing Cost



6.3.3 Cost Minimization: Optimality Conditions I

The two conditions for optimality are:

$$\begin{aligned} q &= F(L, K) && \text{Production equation} \\ MRTS(L, K) &= \frac{w}{r} && \text{Optimality equation.} \end{aligned}$$

This is very similar to the conditions for the consumer's expenditure minimization problem, where the consumer finds the least expensive way to attain utility level u given goods prices p_x and p_y

$$\begin{aligned} u &= u(x, y) && \text{Utility constraint} \\ MRS(x, y) &= \frac{p_x}{p_y} && \text{Optimality equation.} \end{aligned}$$

From a mathematical point of view, the two problems are identical. We use an indifference curve instead of an isoquant, and the absolute slope of the indifference curve, MRS replaces the absolute slope of the isoquant, MRTS. The diagrams are exactly the same.

6.3.4 Cost Minimization: Optimality Conditions II

Recall that

$$\mathrm{MRTS} = \frac{\mathrm{MP_L}}{\mathrm{MP_K}}.$$

This allows us to write the optimality equation in a different way. Now

$$\frac{w}{r} = \mathrm{MRTS} = \frac{\mathrm{MP_L}}{\mathrm{MP_K}}$$

We multiply by $\mathrm{MP_K}$ and divide by w, obtaining

$$\frac{\mathrm{MP_K}}{r} = \frac{\mathrm{MP_L}}{w}.$$

For each input used, the marginal product from a dollar's worth of input must be the same. When we have minimized cost, the marginal product from a dollar's worth of capital is equal to the marginal product from a dollar's worth of labor.

If they were different, we could economize on the more expensive input, use more of the cheaper input, and maintain overall production at a lower cost. We would not be at the cost minimum.

Inputs that are not used have a lower marginal product per dollar of input than those that are used.

This is similar to consumer theory. Recall that in consumer theory we got the very similar condition that

$$\frac{\mathrm{MU}_x}{p_x} = \frac{\mathrm{MU}_y}{p_y}$$

whenever both X and Y are consumed. Goods with a permanently lower marginal utility per dollar were not consumed.

6.3.5 Cobb-Douglas: Marginal Rate of Technical Substitution

We start by computing the marginal rate of technical substitution for the general Cobb-Douglas production function $F(L, K) = AK^aL^b$ with $A, a, b > 0$.

The marginal products are $MP_L = AbK^aL^{b-1}$ and $MP_K = AaK^{a-1}L^b$. The marginal rate of technical substitution is

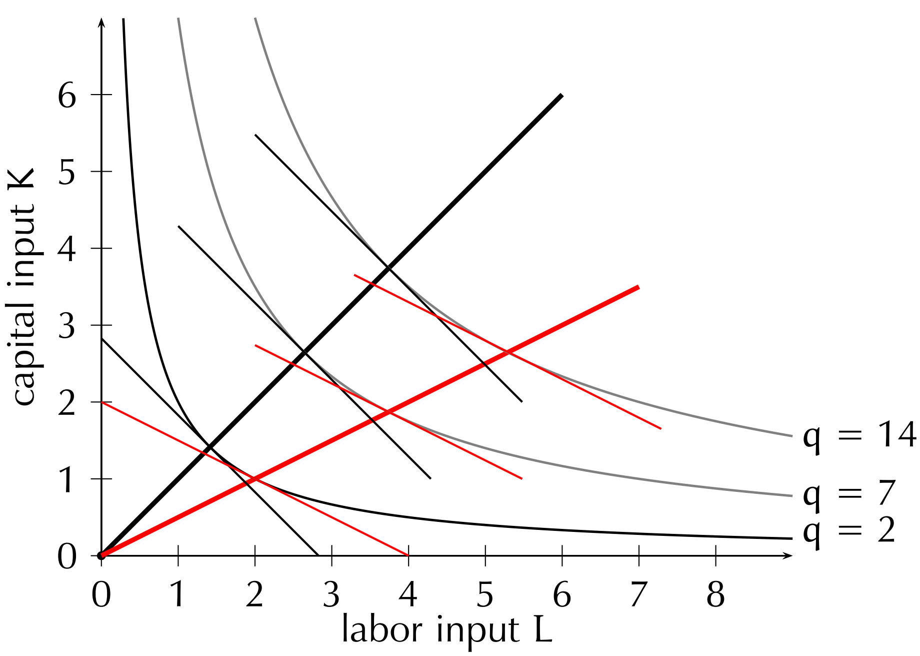
$$\begin{aligned} MRTS &= \frac{MP_L}{MP_K} = \frac{AbK^aL^{b-1}}{AaK^{a-1}L^b} \\ &= \frac{b}{a}\frac{K}{L}. \end{aligned}$$

6.3.6 Cobb-Douglas: Slope and Capital-Labor Ratio

For the Cobb-Douglas family of production functions,

$$\text{MRTS} = \frac{b}{a}\left(\frac{K}{L}\right).$$

This means that the marginal rate of technical substitution depends on capital and labor via the **capital-labor ratio** K/L. Then the slopes of the isoquant must be the same whenever the capital-labor ratio is the same. The absolute slopes, the MRTS's, are constant along every ray thru the origin, a property Cobb-Douglas production shares with a larger family of production functions including linear production functions (perfect substitutes) and Leontief production functions (perfect complements).



6.3.7 Functions with Constant MRTS along Rays

A wide variety of production functions have a constant marginal rate of technical substitution along rays through the origin.

These include the **constant elasticity of substitution** functions

$$F(L, K) = \kappa \left(\alpha L^{-\rho} + (1-\alpha)K^{-\rho}\right)^{-\gamma/\rho}$$

where $-1 < \rho < +\infty$, with $\rho \neq 0$, $\gamma, \kappa > 0$, and $0 < \alpha < 1$. This family of production functions includes several familiar functions. The case $\rho = -1$ and $\gamma = 1$ is linear production. The limit when $\rho \to 0$ is Cobb-Douglas, and letting $\rho \to +\infty$ gives us Leontief production.

These are not the only production functions with a constant MRTS along rays through the origin.

FYI. The **elasticity of substitution** σ is defined by

$$\sigma = -\frac{\partial \ln \mathrm{MRTS}}{\partial \ln(L/K)}$$

6.3.8 Prices and Capital-Labor Ratio

When the marginal rate of technical substitution is constant along rays through the origin, the cost minimizing bundle of inputs will have the same capital-labor ratio as long the input prices remain unchanged. This follows from the fact that

$$\frac{w}{r} = \text{MRTS}\left(\frac{K}{L}\right),$$

so the MRTS remains the same as long as the capital-labor ratio is unchanged. Of course, for many production functions, the MRTS is **not** a function of K/L.

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