Mathematical Economics Final, December 11, 2000

1. Let $f(x,y) = xy$. Is f convex or concave on $\mathbb{R}^2$?

   Answer: Compute the Hessian:
   $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
   The leading principal minors are $H_1 = 0$ and $H_2 = -1$. Since H_2 is non-zero and since this violates the sign pattern for both positive and negative definite, the Hessian is indeterminate. Because a concave C^2 function must have a negative semidefinite matrix, and a convex function must have a positive semidefinite matrix, we can conclude that this function is neither convex nor concave.

   Alternatively, $\frac{1}{4} = f(\frac{1}{2}, \frac{1}{2}) < \frac{1}{2}f(0,0) + \frac{1}{2}f(1,1) = \frac{1}{2}$, so the function is not concave. Also, $\frac{3}{4} = f(\frac{3}{2}, \frac{1}{2}) > \frac{1}{2}f(2,0) + \frac{1}{2}f(1,1) = \frac{1}{2}$, so the function is not convex.

2. Given a price vector of $(3,4)$, find the least-expensive way to obtain exactly 25 utils given a utility function of $u(x,y) = x^{1/3}y^{1/3}$. Be sure to check constraint qualification and the second-order conditions.

   Answer:

   The constraint is $h(x,y) = x^{1/3}y^{1/3} = 25$. Then $Dh = (\frac{1}{3}x^{-2/3}y^{1/3}, \frac{1}{3}x^{1/3}y^{-2/3})$. As this is not $(0,0)$, constraint qualification is satisfied.

   Now form the Lagrangian $\mathcal{L} = 3x + 4y - \lambda(x^{1/3}y^{1/3} - 25)$. The first-order conditions are $3 = (\lambda/3)x^{-2/3}y^{1/3}$, $4 = (\lambda/3)x^{1/3}y^{-2/3}$. Dividing, we find $3/4 = y/x$, or $3x/4 = y$. Substituting in the constraint yields $x^{2/3}(3/4)^{1/3} = 25$. Raising to the 3/2 power gives $\sqrt{3}x/2 = 125$, or $x^* = 250/\sqrt{3}$, and $y^* = 125\sqrt{3}/2$.

   The Hessian of the Lagrangian is:
   $$D^2\mathcal{L} = \frac{\lambda}{9}\begin{bmatrix} 2x^{-5/3}y^{1/3} & x^{-2/3}y^{-2/3} \\ x^{-2/3}y^{-2/3} & -2x^{1/3}y^{-5/3} \end{bmatrix}.$$
   The Hessian itself is positive definite, with determinant $\lambda^2 x^{-2/3}y^{-2/3}/27$ and a positive entry in the upper left. This insures we have a minimum (we don't need to look at the bordered Hessian when the Hessian is positive definite).

3. Find all maxima of $u(x,y) = x + \sqrt{y}$ under the constraints $x \ge 0$, $y \ge 0$, $x + py \le 10$.

   Answer: The Lagrangian is $L(x,y,\mu_0,\mu_1,\mu_2) = x + \sqrt{y} - \mu_0(x + py - 10) + \mu_1 x + \mu_2 y$. The first-order conditions are $1 - \mu_0 + \mu_1 = 0$ and $1/2\sqrt{y} - p\mu_0 + \mu_2 = 0$. The complementary slackness conditions are $\mu_0(x + py - 10) = 0$, $\mu_1 x = 0$, $\mu_2 y = 0$. Of course, $\mu_0, \mu_1, \mu_2 \ge 0$.

   Rewriting the first first-order condition, we find $\mu_0 = 1 + \mu_1 \ge 1$, whic h implies $x + py = 10$ by complementary slackness.

   The second first-order condition can only hold if $y > 0$, which implies $\mu_2 = 0$ by complementary slackness. Thus $1/2\sqrt{y} = p\mu_0 > 0$, so $y = 1/4p^2\mu_0^2$.

   Now if $\mu_0 > 1$, $\mu_1 = \mu_0 - 1 > 0$, so $x = 0$ by complementary slackness. It follows that $py = 10$ $(y = 10/p)$ by the budget constraint, and then $10 = py = 1/4p\mu_0^2$, so $\mu_0 = 1/\sqrt{40p}$. This can only be a solution if $1/\sqrt{40p} < 1$, which is equivalent to $1 > 40p$.

   But if $\mu_0 = 1$, $\mu_1 = 1 - \mu_0 = 0$. The complementary slackness condition for x is satisfied, so $x \ge 0$. Then $y = 1/4p^2$, and $x = 10 - 1/4p \ge 0$. This requires $40p \ge 1$.

   Summing up, if $0 < p < \frac{1}{40}$, $(x,y) = (0,10)$ is the only maximum. If $p = \frac{1}{40}$, $(x,y) = (0,10)$ is still the only maximum. Finally, if $p > \frac{1}{40}$, the maximum is $(10 - \frac{1}{4p}, \frac{1}{4p^2})$.

4. Find and classify all critical points of the function $f(x, y) = (x^2 + y^2)e^{-x^2}$.

Answer: The first-order conditions are $[2x - 2x(x^2 + y^2)]e^{-x^2}$ and $2ye^{-x^2} = 0$. It follows that $y = 0$ and so $2x - 2x^3 = 0$. The critical points are $(1, 0)$, $(0, 0)$, and $(-1, 0)$.

The Hessian is:

$$\begin{bmatrix} [(2 - 6x^2 - 2y^2) - 2x(2x - 2x^3 - 2xy^2)]e^{-x^2} & -4xye^{-x^2} \\ -4xye^{-x^2} & 2e^{-x^2} \end{bmatrix}$$

This yields:

$$H(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad H(1,0) = H(-1,0) = \begin{bmatrix} -4e^{-1} & 0 \\ 0 & 2e^{-1} \end{bmatrix}.$$

It follows that $H(0, 0)$ is positive definite because $H_1 = 2 > 0$ and $H_2 = 4 > 0$. Thus $(0, 0)$ is a local minimum (in fact, it is global). Moreover, $H(1, 0) = H(-1, 0)$ is indefinite because $H_2 = -8e^{-1} < 0$. Thus $(1, 0)$ and $(-1, 0)$ are saddlepoints. The function has no maximum.

5. Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$$

a) What is the rank of A?

Answer: The rank of A is 3. One way to see this is to compute the determinant. It is $(3)(1)(1) + (3)(3)(1) + (3)(2)(0) - (3)(1)(1) - (3)(2)(1) - (3)(3)(0) = 3 + 9 - 3 - 6 = 3$. Since it is non-zero, the matrix is invertible. As it has 3 rows, it has rank 3.

b) How many linearly independent rows does A have? Columns?

Answer: The row rank is the maximum number of linearly independent rows, the column rank is the maximum number of linearly independent columns. Both must be equal to the rank of the matrix. Since the rank is 3, there are 3 linearly independent rows, and 3 linearly independent columns.

c) How many solutions are there to the equation

$$A\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}?$$

Answer: In part (a) we found that A is invertible. It follows that the system of equations has exactly one solution. For the record, the solution is $(4/3, -2, 3)^t$.