Mathematical Economics Final, December 11, 2003

1. Given a prices $p_x = 3$ and $p_y = 4$, minimize expenditure $p_x x + p_y y$ under the constraint $u(x, y) \geq 25$ where $u(x, y) = x^{1/3}y^{1/3}$. Be sure to check constraint qualification and the second-order conditions.

2. Let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

   a) What is the rank of A?

   Answer: The rank of A is 3. One way to see this is to compute the determinant. It is $(3)(1)(1)+(3)(3)(1)+(3)(2)(0)-(3)(1)(1)-(3)(2)(1)-(3)(3)(0) = 3+9-3-6 = 3$. Since it is non-zero, the matrix is invertible. As it has 3 rows, it has rank 3.

   b) How many linearly independent rows does A have? Columns?

   Answer: The row rank is the maximum number of linearly independent rows, the column rank is the maximum number of linearly independent columns. Both must be equal to the rank of the matrix. Since the rank is 3, there are 3 linearly independent rows, and 3 linearly independent columns.

   c) How many solutions are there to the equation $A\mathbf{x} = \mathbf{b}$?

   Answer: In part (a) we found that A is invertible. It follows that the system of equations has exactly one solution. For the record, the solution is $(4/3, -2, 3)^t$.

3. Suppose a firm's production function is $Q = K^{1/3}L^{2/3}$ and that $K = 8000$ and $L = 64$.

   a) How much can the firm produce?

   b) What are the marginal products of capital (K) and labor (L)?

   c) Suppose that the available capital falls by 2 units, while labor increases by 5 units. Without plugging the new numbers for K and L into the production function, compute approximately how much the firm can now produce.

4. Let $f(x, y, z) = x^2 + 3y + z^3 - 5$.

   a) Find an (x_0, y_0, z_0) satisfying $f(x_0, y_0, z_0) = 0$.

   Answer: The point $(1, 1, 1)$ works.

   b) Can x be expressed as a function $g(y, z)$ in some neighborhood of (x_0, y_0, z_0)?

   Answer: Since $\partial f/\partial x = 2x$, $\partial f/\partial x = 2$ at (x_0, y_0, z_0). The Implicit Function Theorem yields such a function g. Alternatively, note that $g(y, z) = (5 - 3y - z^3)^{1/2}$

works.

c) Compute dg.

Answer: By the Implicit Function Theorem, $dg = (-1/2x)(\partial f/\partial y, \partial f/\partial z) = -(3/2)(1, 3z^2)$. At $(1, 1, 1)$, this has the value $(-3/2, -3/2)$.

5. Consider the problem of maximizing $x^2 + y^2$ subject to the constraints $x \geq 1$, $y \geq 2$, and $x + 2y \leq 10$.

   a) Does this problem have a solution?

   b) Is constraint qualification satisfied?

   c) Assuming there is a solution, find it. Make sure to check the second-order conditions.