Mathematical Economics Final, December 7, 2012

1. Let $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$. Find $\sqrt{A}$.

   Answer: The characteristic equation is $\lambda^2 - 6\lambda + 8 = 0$, yielding eigenvalues $\sigma(A) = \{2, 4\}$. The corresponding eigenvectors are $\mathbf{v}_2 = (1, 1)^{\mathbf{T}}$ and $\mathbf{v}_4 = (1, -1)^{\mathbf{T}}$. Let $P = [\mathbf{v}_2, \mathbf{v}_4]$. We have

   $$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

   This implies $D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = P^{-1}AP$. Then $\sqrt{A} = P\sqrt{D}P^{-1}$. Since $\sqrt{D} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 2 \end{pmatrix}$, $\sqrt{A} = \frac{1}{2}\begin{pmatrix} 2+\sqrt{2} & -2+\sqrt{2} \\ -2+\sqrt{2} & 2+\sqrt{2} \end{pmatrix}$.

2. Consider the differential equation $\ddot{x} + 4x = 0$.

   a) Find the solution to the above equation with initial conditions $x(0) = x_0$ and $\dot{x}(0) = x_1$.

   Answer: We find the characteristic equation by substituting $e^{\lambda t}$ for x. This yields $\lambda^2 + 4 = 0$. The eigenvalues are $\pm 2i$. The solution must have the form $\alpha \cos 2t + \beta \sin 2t$. Using the initial conditions we find $\alpha = x_0$ and $\beta = x_1/2$.

   A rather harder approach is to let $y = \dot{x}$. That yields the system

   $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

   The characteristic equation is again $\lambda^2 + 4 = 0$ with eigenvalues $\pm 2i$. Then finding the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, let $P = [\mathbf{v}_1, \mathbf{v}_2]$, and the solution is

   $$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= P^{-1}\begin{pmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{pmatrix} P \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \\ &= P^{-1}\begin{pmatrix} \cos 2t + i\sin 2t & 0 \\ 0 & \cos 2t - i \sin 2t \end{pmatrix} P \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.\end{aligned}$$

   b) Find all steady states.

   Answer: From (a), it is clear that $(x_0, x_1) = (0, 0)$ is the only steady state.

   c) Is this equation Lyapunov stable? Explain.

   Answer: **Yes**. Here $\max\{|\dot{x}(t)|, |x(t)|\} \le x_0^2 + x_1^2$ so by picking (x_0, x_1) small enough, we can guarantee that $(\dot{x}(t), x(t))$ remains within ϵ distance of $(0, 0)$.

d) Is this equation globally asymptotically stable? Explain.

Answer: No. Unless $(x_0, x_1) = (0, 0)$, $x(t)$ will never converge.

3. Let $u_1(x^1, y^1) = x^1 + y^1$ and $u_2(x^2, y^2) = (x^2y^2)^{1/2}$. Suppose $\omega = (5, 5)$ is the endowment, so that $x^i + y^i = 5$ for $i = 1, 2$. Suppose also that $x^i, y^i \geq 0$.

   Is the allocation $(x^1, y^1) = (3, 2)$ and $(x^2, y^2) = (2, 3)$ Pareto optimal? Explain?

   Answer: One method is to note that $u_1(3, 2) = 5$ and $u_2(2, 3) = \sqrt{6}$. Then $u_1(2.5, 2.5) = 5 = u_1(3, 2)$ and $u_(2.5, 2.5) = \sqrt{6.25} > \sqrt{6}$. This shows that $(x^1, y^1) = (2.5, 2.5)$ and $(x^2, y^2) = (2.5, 2.5)$ is a Pareto improvement.

   Alternatively, you can show that the first-order conditions for an interior Pareto optimum require $\text{MRS}^1 = \text{MRS}^2$. Now $\text{MRS}^1 = 1$ at all interior points, but $\text{MRS}^2(x^2, y^2) = y^2/x^2$, which requires $x^2 = y^2$. Since the allocation above does not satisfy this condition, it is not Pareto optimal.

4. On $\mathbb{R}^2_+$, let $u(x_1, x_2) = \sqrt{x_1} + x_2$. Suppose $\mathbf{p} \gg \mathbf{0}$ and $m > 0$. Find the Marshallian demand function $\mathbf{x}(\mathbf{p}, m)$. Don't forget to check constraint qualification and concavity of the objective (or an appropriate second-order condition).

   Answer: The constraints are $g_0(\mathbf{x}) = p_1x_1 + p_2x_2 \leq m$, $g_1(\mathbf{x}) = -x_1 \leq 0$ and $g_3(\mathbf{x}) = -x_2 \leq 0$. The derivative of the constraints is

$$Dg = \begin{pmatrix} p_1 & p_2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   Thus Dg and any 2×2 submatrix have rank 2 while any 1×2 submatrix has rank 1. Since at most two constraints can bind, constraint qualification is satisfied.

   Now

$$D^2u = \begin{pmatrix} -\frac{1}{4}x_1^{-3/2} & 0 \\ 0 & 0 \end{pmatrix}.$$

   This is negative semidefinite for all values of $\mathbf{x}$, so u is concave.

   We now form the Lagrangian $\mathcal{L} = \sqrt{x_1} + x_2 - \lambda(p_1x_1 + p_2x_2 - m) + \mu_1x_1 + \mu_2x_2$. The first order conditions are

$$0 = \frac{1}{2\sqrt{x_1}} - \lambda p_1 + \mu_1$$
$$0 = 1 - \lambda p_2 + \mu_2$$

   The second equation shows $\lambda > 0$, so $p_1x_1 + p_2x_2 = m$ by complementary slackness.

The first equation makes no sense if $x_1 = 0$, so $x_1 > 0$ and $\mu_1 = 0$ by complementary slackness.

There are two cases: $x_2 > 0$ and $x_2 = 0$. If $x_2 > 0$, $\mu_2 = 0$ and we find $x_1 = p_2^2/4p_1^2$ and $x_2 = (m - p_1)/p_2 = m/p_2 - p_2/4p_1$ This requires $4p_1 m > p_2^2$.

The second case is $x_2 = 0$, so $x_1 = m/p_1$. Substituting into the first-order conditions, we find $(1/2)(p_1/m)^{1/2} = \lambda p_1$, so $1 + \mu_2 = \frac{1}{2}p_2\sqrt{1/mp_1}$. Since $\mu_2 \geq 0$, this requires $4p_1 m \leq p_2^2$.

Thus Marshallian demand is

$$\mathbf{x}(\mathbf{p}, m) = \begin{cases} \left(\frac{p_2^2}{4p_1^2}, \frac{4p_1 m - p_2^2}{4p_1 p_2}\right)^{\mathbf{T}} & \text{if } 4p_1 m > p_2^2 \\ \left(\frac{m}{p_1}, 0\right)^{\mathbf{T}} & \text{if } 4p_1 m \leq p_2^2 \end{cases}.$$

5. Consider the set $S = \{\mathbf{x} \in \mathbb{R}^2 : 1 \leq \|\mathbf{x}\| < 2\}$.

   a) Is the set S closed? Open? Explain

   Answer: The set S is neither open nor closed. It is not open because any ϵ-ball containing $(1, 0)$ will contain $(1 - \epsilon/2, 0$, which is not in S. It is not closed because $\mathbf{x}^n = (2 - 1/n, 0) \in S$, but $\mathbf{x}^n \to (2, 0) \notin S$.

   b) Is the set S compact? Explain.

   Answer: No. Compact sets in $\mathbb{R}^2$ must be closed and bounded. By part (a) S is not closed, and so cannot be compact.

   c) Is the set S convex? Explain.

   Answer: The line segment between $(1, 0)$ and $(-1, 0)$, which are both points of S contains $(0, 0) = \frac{1}{2}(1, 0) + \frac{1}{2}(-1, 0) \notin S$.