# Mathematical Economics Exam #2, November 4, 2014

1. Consider the function $\mathbf{f}(t) = \begin{bmatrix} t^2 \\ -t \end{bmatrix}$.

   *a*) Compute the tangent vector of $\mathbf{f}$ at any $t$.

   *b*) Give an equation for the tangent line at the point $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$.

   **Answer:**

   *a*) The tangent vector is given by the derivative

   $$\frac{d\mathbf{f}}{dt} = \begin{bmatrix} 2t \\ -1 \end{bmatrix}.$$

   *b*) The point is $\mathbf{f}(2)$, so the tangent vector is $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$. There are several way to write the tangent line. One is that

   $$L = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} + t \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right\}.$$

   It can also be written by eliminating $t$ from the equations. For that, $y = -2-t$, so $t = -(2+y)$. Substituting in $x = 4 + 4t$ yields $-4 = x + 4y$.

2. Let $f\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ be defined by $f(x, y) = x^{2/3}y^{1/3}$.

   *a*) Find the level curves of $f$.

   *b*) Use the Implicit Function Theorem to show that $y(x)$ defined by $f\big(x, y(x)\big) = q$ for $q > 0$ defines a $C^1$ function $y$.

   *c*) Using $y$ as in part (b), compute $dy/dx$.

   *d*) What happens if $q = 0$ in part (b). In particular, can you still compute $dy/dx$?

   **Answer:**

   *a*) The level curves are $\{(x, y) \in \mathbb{R}^2_+ : x^{2/3}y^{1/3} = q\}$ for $q \geq 0$.

   *b*) We examine whether $\frac{\partial f}{\partial y} = (1/3)x^{2/3}y^{-2/3} \neq 0$ for $x, y \neq 0$. This condition is satisfied since $x^{2/3}y^{1/3} = q > 0$. Because the partial derivative is non-zero, the Implicit Function Theorem tells us $y(x)$ is a $C^1$ function.

   *c*) Using the Implicit Function Theorem, we find

   $$\frac{dy}{dx} = -\left(\frac{1}{3}\frac{x^{2/3}}{y^{2/3}}\right)^{-1} \times \left(\frac{2}{3}\frac{y^{1/3}}{x^{1/3}}\right) = -\frac{2y}{x}.$$

*d*) If $q = 0$, the level curve consists of both the non-negative $x$-axis and non-negative $y$-axis. When $x > 0$, $y(x) = 0$, and when $x = 0$, $y(x)$ can be any non-negative number. When $x > 0$, $dy/dx = 0$, and when $x = 0$, $dy/dx$ is undefined.

3. Consider the problem of maximizing $3x + 4y$ subject to the constraint that $x^2 + y \leq 5$, $x \geq 0$ and $y \geq 0$.

   *a*) Without calculating it, prove this problem has a solution.

   *b*) Find the solution. Don't forget to check constraint qualification.

**Answer:**

*a*) Since the constraint set is compact (closed and bounded) and $3x + 4y$ is continuous, the Weierstrass Theorem guarantees there is a solution.

*b*) It is obvious that all three constraints cannot simultaneously bind. We consider the matrix

$$dg = \begin{bmatrix} 2x & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

All rows are non-zero, and any two rows are linearly independent provided $x > 0$. When $x = 0$, we must include the second row when either other row will give us a linearly independent pair. This implies that constraint qualification holds.

Now form the Lagrangian $\mathcal{L} = 3x+4y-\lambda(x^2+y-5)+\mu_x x+\mu_y y$. The first-order conditions are

$$3 = 2\lambda x - \mu_x, \text{ and} \tag{1}$$

$$4 = \lambda - \mu_y. \tag{2}$$

Here $\lambda \geq 4 + \mu_y \geq 4 > 0$, so $x^2 + y = 5$ by complementary slackness. There are three cases to consider.

If $x = 0$, then (1) becomes $3 = -\mu_x \leq 0$, which is impossible.

If $y = 0$, then $x = \sqrt{5}$ and $\mu_x = 0$ by complementary slackness. It follows that $\lambda = 3/2\sqrt{5}$ by (1), which contradicts (2).

The only possibility left is $x > 0$ and $y > 0$. Then $\mu_x = \mu_y = 0$ by complementary slackness. This implies $3 = 2\lambda x$ and $\lambda = 4$. Thus $x = 3/8$, so

$y = 5 - (3/8)^2 = 311/64$. As the only remaining option, $(3/8, 311/64)$ must be the maximum.

4. Consider the quadratic form $Q(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + x_2^2 + 3x_1x_3 - x_3^2$ with constraint $x_2 + x_3 = 0$. Does this problem have a maximum, minimum, or saddlepoint at $(0, 0, 0)$? Explain why.

   **Answer:** We form the bordered Hessian

$$H = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3/2 \\ 1 & 1 & 1 & 0 \\ 1 & 3/2 & 0 & -1 \end{bmatrix}.$$

   There are three variables ($n = 3$) and one linear constraint ($m = 1$), so we look at the last $n - m = 2$ leading principal minors. They are $H_3 = -1$ and $H_4 = +1/4$. Since $H_4(-1)^n = H_4(-1)^m = -1/4 < 0$, the quadratic form fails both the tests for positive definiteness and negative definiteness on the constraint set. As $H_4$ is non-zero, we may conclude that H is indefinite on the constraint set and that $(0, 0, 0)$ is a saddlepoint.

   Alternatively, consider the points $\mathbf{x}_1 = \varepsilon(1, 2, -2)$ and $\mathbf{x}_2 = \varepsilon(1, -2, 2)$. Both satisfiy the constraint and $Q(\mathbf{x}_1) = -\varepsilon^2$ and $Q(\mathbf{x}_2) = 3\varepsilon^2$. This shows that Q takes both positive and negative values in any neighborhood of $\mathbf{0}$, so $\mathbf{0}$ is neither a constrained local max, nor constrained local min.