Mathematical Economics Exam #1, September 27, 2016

1. Consider the vector $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. The collection $\mathscr{B}$ is defined by

$$\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \right\}.$$

   a) Show that $\mathscr{B}$ is a basis.

   b) Find the coordinates of $\mathbf{x}$ in the basis $\mathscr{B}$.

Answer:

a) We first form the matrix

$$B = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3] = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$$

It is clear this matrix is invertible as its determinant is 5. It follows that $\mathscr{B}$ is a basis.

b) There are two ways we can find the coordinates $\mathbf{z}$ of $\mathbf{x}$ in the $\mathscr{B}$ basis. The shortest is to solve the equation $\mathbf{x} = B\mathbf{z}$.

The other method is to compute B^{-1}, when $\mathbf{z} = B^{-1}\mathbf{x}$.

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 5 & -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/5 & 1/5 & 1/5 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3/5 & 2/5 & -3/5 \\ 0 & 0 & 1 & -1/5 & 1/5 & 1/5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1/5 & -4/5 & 6/5 \\ 0 & 1 & 0 & 3/5 & 2/5 & -3/5 \\ 0 & 0 & 1 & -1/5 & 1/5 & 1/5 \end{pmatrix}.$$

It follows that

$$B^{-1} = \frac{1}{5} \begin{pmatrix} -1 & -4 & 6 \\ 3 & 2 & -3 \\ -1 & 1 & 1 \end{pmatrix}.$$

The coordinates $\mathbf{z}$ of $\mathbf{x}$ in the basis $\mathscr{B}$ are $B^{-1}\mathbf{x} = \begin{pmatrix} -3/5 \\ 4/5 \\ 2/5 \end{pmatrix}$.

2. Consider the sequence defined by $x_n = (-1)^n + 1/n$.

   a) Does this sequence converge? Explain why or why not?

   b) Find a convergent subsequence of $\{x_n\}_{n=0}^{\infty}$

   Answer:

   a) Here $|x_n - x_{n+1}| = |(-1)^n + (-1)^{n+1} + 1/n - 1/(n+1)| = |(-1)^n + (-1)^{n+1} + 1/n(n+1)| \to 2$. Now suppose $x_n \to x$. Choose $\varepsilon = 1/2$ and find N large enough that $|x_n - x| < 1/2$ for $n > N$. Then $|x_n - x_{n+1}| \leq |x_n - x| + |x - x_{n+1}| < 1$ for $n > N$. This contradicts the fact that the left-hand side converges to 2.

   b) The subsequence $x_{n_i} = x_{2i} = 1 + 1/(2i)$ converges to 1.

3. Consider the linear system

$$\begin{aligned} w + x + y + z &= 1 \\ w + 2x + 3y + 4z &= 1 \\ w + 4x + 9y + 16z &= 1. \end{aligned}$$

   a) How many solutions does the system have?

   b) Find all solutions to the system.

   Answer: We start by row-reducing the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 1 \\ 1 & 4 & 9 & 16 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 3 & 8 & 15 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 2 & 6 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{pmatrix}$$

   a) It is clear that the rank of the coefficient and augmented matrices are both 3. It follows that the system can be solved. Moreover, since there is one free variable, there are infinitely many solutions.

   b) The solutions may be written $w = 1 - z$, $x = 3z$, $y = -3z$, where z is any real number.

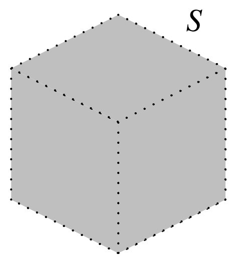
4. Consider $S = \{(x_1, x_2, x_3) : -1 < x_i < +1 \text{ for } i = 1, 2, 3\}$.

   a) Sketch the set S.

   b) Using the Euclidean norm on $\mathbb{R}^3$, determine whether S is a closed set, open set, both, or neither. Justify your answer.

Answer:

a) The set S is an open cube with side 2, centered on the origin.



b) Here S is an open set. Let $\mathbf{x} = (x_1, x_2, x_3) \in S$ and define $\varepsilon = \min\{|1-x_i|, |-1-x_i| : i = 1, 2, 3\}$. Then whenever $\mathbf{y} \in B_\varepsilon(\mathbf{x})$, $\mathbf{y} \in S$. Since S contains an open ball about each of its points, S is open.

The set S is not closed since $\mathbf{x}_n = (1 - 1/n, 0, 0) \in S$ for every $n = 1, 2, 3, \ldots$, but $\lim_n \mathbf{x}_n = (1, 0, 0) \notin S$.