Mathematical Economics Exam #2, November 1, 2016

1. Consider the problem of maximizing \( u(x, y) = x + \sqrt{y} \) subject to the constraints \( x, y \geq 0 \) and \( px + y \leq 10 \) where \( p > 0 \).

   a) Is constraint qualification satisfied?

   **Answer:** The derivative of the constraints is:

   \[
   \begin{pmatrix}
   p & 1 \\
   -1 & 0 \\
   0 & -1 \\
   \end{pmatrix}
   \]

   At most two of the constraints can bind. Since all rows are non-zero and any two rows are independent, the rank will equal the number of binding constraints. NDCQ is satisfied.

   b) Find the solution to the maximization problem.

   **Answer:** The Lagrangian is \( \mathcal{L} = x + \sqrt{y} - \lambda_0(px + y - 10) + \lambda_1 x + \lambda_2 y \). The first-order conditions are:

   \[
   \begin{align*}
   0 &= 1 - p\lambda_0 + \lambda_1 \quad (1) \\
   0 &= \frac{1}{2\sqrt{y}} - \lambda_0 + \lambda_2 \quad (2)
   \end{align*}
   \]

   Equation (1) tells us \( p\lambda_0 = 1 + \lambda_1 \geq 1 \). Then \( \lambda_0 > 0 \) and it follows that \( px + y = 10 \) by complementary slackness.

   There are now three cases to consider: 1) \( x = 0, y = 10 \); 2) \( x = 10/p, y = 0 \); 3) \( x, y > 0 \).

   In case (1), \( \lambda_2 = 0 \) by complementary slackness, so \( \lambda_0 = 1/2\sqrt{10} \). This requires \( p/2\sqrt{10} \geq 1 \), that is, \( p^2 \geq 40 \).

   In case (2), equation (2) is violated due to division by zero. There is no solution here.

   In case (3), \( \lambda_1 = \lambda_2 = 0 \) by complementary slackness. It follows that \( p\lambda_0 = 1 \). Then equation (2) becomes \( \frac{1}{2}y^{-1/2} = 1/p \), so \( y = p^2/4 \). This works provided \( p^2 \leq 40 \). In that case, \( x = (10 - p^2/4)/p = (40 - p^2)/4p \).

   In sum, the solution is:

   \[
   \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ 10 \end{pmatrix} & \text{for } p^2 \geq 40 \\ \begin{pmatrix} (40 - p^2)/4p \\ p^2/4 \end{pmatrix} & \text{for } p^2 \leq 40 \end{cases}
   \]
Notice that both cases agree when $p^2 = 40$.

2. Let $f(x, y, z) = x^2 + 5y + z^3$.
   
   a) Does this function map $\mathbb{R}^3$ onto $\mathbb{R}$?
   
   **Answer:** Yes, it is onto. In fact, we can even set two of the variables to zero. Here $f(0, w/5, 0) = w$, showing that the function takes all real values.

   b) Find a point $(x_0, y_0, z_0)$ satisfying $f(x_0, y_0, z_0) = 7$.

   **Answer:** The point $(1, 1, 1)$ works. As do $(7^{1/2}, 0, 0)$, $(0, 7/5, 0)$ and $(0, 0, 7^{1/3})$. The IFT does not apply in cases where $x = 0$, but does apply when $x \neq 0$.

   c) Given your choice $(x_0, y_0, z_0)$, is there a differentiable function $g(y, z)$ on some neighborhood of $(y_0, z_0)$ that obeys $x_0 = g(y_0, z_0)$ and $f(g(y, z), y, z) = 7$?

   **Answer:** Since $\frac{\partial f}{\partial x} = 2x$, $(\frac{\partial f}{\partial x})(1, 1, 1) = 2$ The Implicit Function Theorem yields such a function $g$. Alternatively, note that $g(y, z) = +(7 - 5y - z^3)^{1/2}$ works.

   d) Compute $dg$.

   **Answer:** By the Implicit Function Theorem,

   $$dg = -\frac{1}{2x} \left( \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = -\frac{1}{2}(5, 3z^2).$$

   At $(1, 1, 1)$, this has the value $(-5/2, -3/2)^T$.

3. Let $f(x, y) = xy^2 + x^3y - xy$. Find all critical points of $f$ and classify them (local max, local min, saddlepoint, other/unknown).

   **Answer:** The first-order conditions are

   $$0 = y^2 + 3x^2y - y$$
   $$0 = 2xy + x^3 - x$$

   There are six critical points. They are: $x_0 = (0, 0)$, $x_1 = (0, 1)$, $x_2 = (1, 0)$, $x_3 = (-1, 0)$, $x_4 = (1/\sqrt{5}, 2/5)$, and $x_5 = (-1/\sqrt{5}, 2/5)$.

   We now consider the Hessian

   $$H = \frac{\partial^2 f}{\partial x \partial y} = \begin{pmatrix}
   6xy & 3x^2 + 2y - 1 \\
   3x^2 + 2y - 1 & 2x
   \end{pmatrix}.$$

   At $x_0$ and $x_1$, $H_1 = 0$ and $H_2 = -1$. It follows that we have a saddlepoint. At $x_2$ and $x_3$, $H_1 = 0$ and $H_2 = -4 < 0$. Once again, we have a saddlepoint. At $x_4$, $H_1 = 12/5\sqrt{5} > 0$ and
H_2 = 36x^2y^2 = 144/625 > 0. The Hessian is positive definite and we have a **local minimum**. At x_5, H_1 = -12/5\sqrt{5} < 0 and H_2 = 36x^2y^2 = 144/625 > 0. The Hessian is negative definite and we have a **local maximum**.

4. Consider the quadratic form Q(x, y, z) = x^2 + 4xy - 2y^2 + 6yz with constraint x + y + z = 0.

   a) Find a symmetric matrix that defines this quadratic form.

   **Answer:** The matrix that defines the quadratic form is
   
   \[
   A = \begin{pmatrix}
   1 & 2 & 0 \\
   2 & -2 & 3 \\
   0 & 3 & 0
   \end{pmatrix}
   \]

   b) Use the bordered Hessian to determine whether the quadratic form has a constrained maximum, minimum, or saddlepoint at (0, 0, 0)?

   **Answer:** We form the bordered Hessian

   \[
   H = \begin{pmatrix}
   0 & 1 & 1 & 1 \\
   1 & 1 & 2 & 0 \\
   1 & 2 & -2 & 3 \\
   1 & 0 & 3 & 0
   \end{pmatrix}
   \]

   There are 3 variables (n = 3) and one linear constraint (m = 1), so we look at the last \( n - m = 2 \) leading principal minors. They are \( H_3 = +5 \) and \( H_4 = +9 \). Since \( (-1)^m = (-1)^n = -1 \), which has a different sign than \( H_4 = +9 \), the quadratic form fails both the tests for positive definiteness and negative definiteness on the constraint set. As \( H_4 \) is non-zero, we may conclude that \( H \) is indefinite on the constraint set and that (0, 0, 0) is a saddlepoint.