Mathematical Economics Exam #2, November 1, 2016

1. Consider the problem of maximizing $u(x, y) = x + \sqrt{y}$ subject to the constraints $x, y \geq 0$ and $px + y \leq 10$ where $p > 0$.

   a) Is constraint qualification satisfied?

   Answer: The derivative of the constraints is:

$$\begin{pmatrix} p & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

   At most two of the constraints can bind. Since all rows are non-zero and any two rows are independent, the rank will equal the number of binding constraints. NDCQ is satisfied.

   b) Find the solution to the maximization problem.

   Answer: The Lagrangian is $\mathcal{L} = x + \sqrt{y} - \lambda_0(px + y - 10) + \lambda_1 x + \lambda_2 y$. The first-order conditions are:

$$0 = 1 - p\lambda_0 + \lambda_1 \tag{1}$$

$$0 = \frac{1}{2\sqrt{y}} - \lambda_0 + \lambda_2 \tag{2}$$

   Equation (1) tells us $p\lambda_0 = 1 + \lambda_1 \geq 1$. Then $\lambda_0 > 0$ and it follows that $px + y = 10$ by complementary slackness.

   There are now three cases to consider: 1) $x = 0$, $y = 10$; 2) $x = 10/p$, $y = 0$; 3) $x, y > 0$.

   In case (1), $\lambda_2 = 0$ by complementary slackness, so $\lambda_0 = 1/2\sqrt{10}$. This requires $p/2\sqrt{10} \geq 1$, that is, $p^2 \geq 40$.

   In case (2), equation (2) is violated due to division by zero. There is no solution here.

   In case (3), $\lambda_1 = \lambda_2 = 0$ by complementary slackness. It follows that $p\lambda_0 = 1$. Then equation (2) becomes $\frac{1}{2}y^{-1/2} = 1/p$, so $y = p^2/4$. This works provided $p^2 \leq 40$. In that case, $x = (10 - p^2/4)/p = (40 - p^2)/4p$.

   In sum, the solution is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ 10 \end{pmatrix} & \text{for } p^2 \geq 40 \\ \begin{pmatrix} (40 - p^2)/40p \\ p^2/4 \end{pmatrix} & \text{for } p^2 \leq 40. \end{cases}$$

Notice that both cases agree when $p^2 = 40$.

2. Let $f(x, y, z) = x^2 + 5y + z^3$.

   a) Does this function map $\mathbb{R}^3$ onto $\mathbb{R}$?

   Answer: Yes, it is onto. In fact, we can even set two of the variables to zero. Here $f(0, w/5, 0) = w$, showing that the function takes all real values.

   b) Find a point (x_0, y_0, z_0) satisfying $f(x_0, y_0, z_0) = 7$.

   Answer: The point $(1, 1, 1)$ works. As do $(7^{1/2}, 0, 0)$, $(0, 7/5, 0)$ and $(0, 0, 7^{1/3})$. The IFT does not apply in cases where $x = 0$, but does apply when $x \neq 0$.

   c) Given your choice (x_0, y_0, z_0), is there a differentiable function $g(y, z)$ on some neighborhood of (y_0, z_0) that obeys $x_0 = g(y_0, z_0)$ and $f\big(g(y, z), y, z\big) = 7$?

   Answer: Since $\partial f/\partial x = 2x$, $(\partial f/\partial x)(1, 1, 1) = 2$ The Implicit Function Theorem yields such a function g. Alternatively, note that $g(y, z) = +(7 - 5y - z^3)^{1/2}$ works.

   d) Compute dg.

   Answer: By the Implicit Function Theorem,

$$dg = -\frac{1}{2x}\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = -\frac{1}{2}(5, 3z^2).$$

   At $(1, 1, 1)$, this has the value $(-5/2, -3/2)^T$.

3. Let $f(x, y) = xy^2 + x^3y - xy$. Find all critical points of f and classify them (local max, local min, saddlepoint, other/unknown).

   Answer: The first-order conditions are

$$\begin{aligned} 0 &= y^2 + 3x^2y - y \\ 0 &= 2xy + x^3 - x \end{aligned}$$

There are six critical points. They are: $x_0 = (0, 0)$, $x_1 = (0, 1)$, $x_2 = (1, 0)$, $x_3 = (-1, 0)$, $x_4 = (1/\sqrt{5}, 2/5)$, and $x_5 = (-1/\sqrt{5}, 2/5)$.

We now consider the Hessian

$$H = d^2f = \begin{pmatrix} 6xy & 3x^2 + 2y - 1 \\ 3x^2 + 2y - 1 & 2x \end{pmatrix}.$$

At x_0 and x_1, $H_1 = 0$ and $H_2 = -1$. It follows that we have a **saddlepoint**. At x_2 and x_3, $H_1 = 0$ and $H_2 = -4 < 0$. Once again, we have a **saddlepoint**. At x_4, $H_1 = 12/5\sqrt{5} > 0$ and

$H_2 = 36x^2y^2 = 144/625 > 0$. The Hessian is positive definite and we have a **local minimum**. At x_5, $H_1 = -12/5\sqrt{5} < 0$ and $H_2 = 36x^2y^2 = 144/625 > 0$. The Hessian is negative definite and we have a **local maximum**.

4. Consider the quadratic form $Q(x, y, z) = x^2 + 4xy - 2y^2 + 6yz$ with constraint $x + y + z = 0$.

   a) Find a symmetric matrix that defines this quadratic form.

   Answer: The matrix that defines the quadratic form is

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -2 & 3 \\ 0 & 3 & 0 \end{pmatrix}.$$

   b) Use the bordered Hessian to determine whether the quadratic form has a constrained maximum, minimum, or saddlepoint at $(0, 0, 0)$?

   Answer: We form the bordered Hessian

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & -2 & 3 \\ 1 & 0 & 3 & 0 \end{pmatrix}.$$

   There are 3 variables ($n = 3$) and one linear constraint ($m = 1$), so we look at the last $n - m = 2$ leading principal minors. They are $H_3 = +5$ and $H_4 = +9$. Since $(-1)^m = (-1)^n = -1$, which has a different sign that $H_4 = +9$, the quadratic form fails both the tests for positive definiteness and negative definiteness on the constraint set. As H_4 is non-zero, we may conclude that H is indefinite on the constraint set and that $(0, 0, 0)$ is a saddlepoint.