1. Consider the differential equation \( \ddot{y} - 6\dot{y} + 9y = 0 \).

   a) Find the general solution.
   b) The point \( y = 0 \) is a steady state. Is it Lyapunov stable?
   c) Find the solution that obeys \( y(0) = 1, \dot{y}(0) = 2 \).

   Answer:
   a) The characteristic equation is \( \lambda^2 - 6\lambda + 9 = 0 \). The only solution is \( \lambda = 3 \). In such a case, the general real-valued solution is 
   \[
   y(t) = \alpha e^{3t} + \beta te^{3t}.
   \]
   b) No. It is not Lyapunov stable. For any \((\alpha, \beta) \neq (0, 0)\), the solution obeys \( |y(t)| \to \infty \), showing that the steady state \( y = 0 \) is not stable in any sense.
   c) Now \( 1 = y(0) = \alpha \) and \( 2 = \dot{y}(0) = 3\alpha + \beta \) so \( \beta = -1 \). The solution obeying \( y(0) = 1 \) and \( \dot{y}(0) = 2 \) is 
   \[
   y(t) = e^{3t} - te^{3t}.
   \]

2. Consider the function \( f(x) = e^{-x} \) defined on \( \mathbb{R} \).
   a) Is \( f \) quasiconvex?
   b) Is \( f \) quasiconcave?
   c) Is \( f \) convex?
   d) Is \( f \) concave?

   Answer:
   a) Yes. Here \( f' < 0 \) and all monotonic functions on \( \mathbb{R} \) are quasiconvex.
   b) Yes. Here \( f' < 0 \) and all monotonic functions on \( \mathbb{R} \) are quasiconcave.
   c) Yes. Here \( f'' = e^{-x} > 0 \), showing that \( f \) is strictly convex.
   d) No. Since \( f \) is strictly convex, it cannot be concave.

3. Consider the problem of maximizing the function \( u(x, y) = x - e^{-y} \) subject to the constraint \( x + 3y \leq 30 \) and the non-negativity constraints \( x \geq 0, y \geq 0 \).
   a) Does this problem have a solution? Explain?
b) If the problem has a solution, use the Kuhn-Tucker theorem to find it. Don’t forget to check constraint qualification and the second-order conditions.

Answer:

a) Yes, it has a solution. We are trying to maximize a continuous function over a compact budget set (we showed in class that such sets are compact). The Weierstrass Theorem tells us there is a solution.

b) There are three constraints, but at most two can bind at any time. The derivative of the constraints is

\[ dg = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Any pair of constraints yields a matrix of rank 2, while a single constraint gives a matrix of rank 1. It follows that constraint qualification (NDCQ) is satisfied.

Note that the Hessian of \( x - e^{-y} \) is

\[ H = \begin{pmatrix} 0 & 0 \\ 0 & -e^{-y} \end{pmatrix}. \]

This is always negative semidefinite, so the objective is concave. It follows that solutions to the first-order conditions will be maxima.

The Lagrangian is \( \mathcal{L} = x - e^{-y} - \lambda(x + 3y - 30) + \mu_x x + \mu_y y \). The first-order conditions are:

\[
\frac{\partial \mathcal{L}}{\partial x} = 1 - \lambda + \mu_x = 0 \\
\frac{\partial \mathcal{L}}{\partial y} = e^{-y} - 3\lambda + \mu_y = 0.
\]

**Case I:** \( x, y > 0 \). Here \( \mu_x = \mu_y = 0 \) by complementary slackness. Then \( \lambda = 1 \) and \( e^{-y} = 3 \). It follows that \( y = -\ln 3 < 0 \). This violates the constraint that \( y \geq 0 \), so there is no solution with \( x, y > 0 \).

**Case II:** \( x = 0 \). Then \( \lambda = 1 + \mu_x \geq 1 \), so the budget constraint binds. It follows that \( y = 10 \). Then \( e^{-10} + \mu_y = 3\lambda \geq 3 \). This requires \( \mu_y > 0 \) when \( y = 0 \) by complementary slackness. This contradiction means there is no solution with \( x = 0 \).

**Case III:** \( x > 0, y = 0 \). Then \( \lambda = 1 \) and so \( \mu_y = 2 \). This satisfies the first order conditions. Since \( \lambda = 1 \), the budget constraint binds by complementary slackness, implying \( x = 30 \).
Case IV: \( x = 0, y = 0 \). Then \( \lambda = 0 \) by complementary slackness. By the first-order conditions, \( \mu_x = -1 \), which is impossible. This is not a solution.

It follows that the maximum is at \((30, 0)\), where \( u(30, 0) = 29 \).

4. Consider the difference equation

\[
x_{t+1} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} x_t.
\]

a) Find the eigenvalues of the system.

b) Find eigenvectors corresponding to the eigenvalues.

c) Do the eigenvectors form a basis for \( \mathbb{R}^2 \)? Explain.

d) Write the general solution of this difference equation.

Answer:

a) The characteristic equation is \( \lambda^2 - 4 = 0 \). The eigenvalues are \( \lambda = \pm 2 \).

b) For \( \lambda = 2 \), the eigenvectors must obey

\[
\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} v = 0.
\]

This is satisfied by any non-zero multiple of \( v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \).

For \( \lambda = -2 \), the eigenvectors must obey

\[
\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} v = 0.
\]

This is satisfied by any non-zero multiple of \( v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \).

c) Since the eigenvalues are distinct, the eigenvectors must form a basis. Alternatively, the fact that \( \det[v_1, v_2] = -4 \), which is non-zero, shows that we have two linearly independent vectors which must form a basis for \( \mathbb{R}^2 \).

d) The general solution is \( x_t = c_1 2^t v_1 + c_2 (-2)^t v_2 \) or

\[
x_t = c_1 2^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 (-2)^t \begin{pmatrix} 2 \\ -1 \end{pmatrix}.
\]
5. Find and classify all critical points of the function $u(x, y) = x^2 + 2xy + 2y^3$ subject to the constraint $x + 3y \leq 3$. Don’t forget to check constraint qualification and the second-order conditions.

**Answer:** We rewrite the constraint $g(x, y) = 3 - x - 3y \geq 0$. Then $dg = -(1, 3)$. Thus $dg$ has maximum rank (i.e, rank one), satisfying constraint qualification. (We don’t have to worry about the case where $g$ doesn’t bind.) Alternatively, the constraint is linear, so constraint qualification is satisfied.

The Lagrangian is $\mathcal{L} = x^2 + 2xy + 2y^3 + \lambda(3 - x - 3y)$. The multiplier $\lambda$ will obey $\lambda \geq 0$ for a maximum and $\lambda \leq 0$ for a minimum. The first-order conditions are:

\[
\begin{align*}
0 &= 2x + 2y - \lambda \\
0 &= 2x + 6y^2 - 3\lambda.
\end{align*}
\]

If the constraint does not bind (i.e., $x + 3y < 3$), $\lambda = 0$ by complementary slackness. The first equation then tells us $y = -x$ and the second yields $y = 0$ or $y = 1/3$. Both solutions satisfy the constraint yielding critical points of $(0, 0)$ and $(-1/3, 1/3)$. The Hessian is then

\[
H = \begin{pmatrix}
2 & 2 \\
2 & 12y
\end{pmatrix}.
\]

We check the leading principal minors. At $(0, 0)$, $|H_1| = 2 > 0$ and $|H_2| = -4 < 0$, so it is neither a maximum nor minimum. At $(-1/3, 1/3)$, $|H_1| = 2 > 0$ and $|H_2| = 4 > 0$, so it is a local minimum.

This leave the case where the constraint binds, where $x = 3 - 3y$. Substituting in the first-order conditions we find

\[
\begin{align*}
\lambda &= 6 - 4y \\
\lambda &= 2 - 2y + 2y^2.
\end{align*}
\]

Eliminating $\lambda$, we find $2y^2 + 2y - 4 = 0$. This has solutions $y = 1$ and $y = -2$. The corresponding critical points are $(0, 1)$ and $(9, -2)$, both with $\lambda > 0$. Here we must consider the bordered Hessian

\[
B = \begin{pmatrix}
0 & -1 & -3 \\
-1 & 2 & 2 \\
-3 & 2 & 12y
\end{pmatrix}.
\]
With two variables and one constraint, we only look at the determinant \( \det B = -6 - 12y \). This is negative when \( y = 2 \) (local minimum) and positive when \( y = -1 \) (local maximum).

I did not ask about global maxima or minima, but \( \lim_{t \to \pm \infty} f(0, y) = \pm \infty \), so it has neither a global maximum nor global minimum.

In summation, there are four critical points: \((0, 0), (-1/3, 1/3), (0, 1), \text{ and } (9, -2)\). The first is a saddlepoint, the second and fourth local minima, and the third a local maximum.