Mathematical Economics Final, December 12, 2017

1. Consider the differential equation $\ddot{y} - 6\dot{y} + 9y = 0$.

   a) Find the general solution.

   b) The point $y = 0$ is a steady state. Is it Lyapunov stable?

   c) Find the solution that obeys $y(0) = 1$, $\dot{y}(0) = 2$.

   Answer:

   a) The characteristic equation is $\lambda^2 - 6\lambda + 9 = 0$. The only solution is $\lambda = 3$. In such a case, the general real-valued solution is

   $$y(t) = \alpha e^{3t} + \beta t e^{3t}.$$

   b) No. It is **not** Lyapunov stable. For any $(\alpha, \beta) \neq (0, 0)$, the solution obeys $|y(t)| \to \infty$, showing that the steady state $y = 0$ is **not stable** in any sense.

   c) Now $1 = y(0) = \alpha$ and $2 = \dot{y}(0) = 3\alpha + \beta$ so $\beta = -1$. The solution obeying $y(0) = 1$ and $\dot{y}(0) = 2$ is

   $$y(t) = e^{3t} - te^{3t}.$$

2. Consider the function $f(x) = e^{-x}$ defined on $\mathbb{R}$.

   a) Is f quasiconvex?

   b) Is f quasiconcave?

   c) Is f convex?

   d) Is f concave?

   Answer:

   a) **Yes**. Here $f' < 0$ and all monotonic functions on $\mathbb{R}$ are quasiconvex.

   b) **Yes**. Here $f' < 0$ and all monotonic functions on $\mathbb{R}$ are quasiconcave.

   c) **Yes**. Here $f'' = e^{-x} > 0$, showing that f is strictly convex.

   d) **No**. Since f is stricly convex, it cannot be concave.

3. Consider the problem of maximizing the function $u(x, y) = x - e^{-y}$ subject to the constraint $x + 3y \leq 30$ and the non-negativity constraints $x \geq 0$, $y \geq 0$.

   a) Does this problem have a solution? Explain?

b) If the problem has a solution, use the Kuhn-Tucker theorem to find it. Don't forget to check constraint qualification and the second-order conditions.

Answer:

a) Yes, it has a solution. We are trying to maximize a continuous function over a compact budget set (we showed in class that such sets are compact). The Weierstrass Theorem tells us there is a solution.

b) There are three constraints, but at most two can bind at any time. The derivative of the constraints is

$$d\mathbf{g} = \begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Any pair of constraints yields a matrix of rank 2, while a single constraint gives a matrix of rank 1. It follows that constraint qualification (NDCQ) is satisfied.

Note that the Hessian of $x - e^{-y}$ is

$$\mathbf{H} = \begin{pmatrix} 0 & 0 \\ 0 & -e^{-y} \end{pmatrix}.$$

This is always negative semidefinite, so the objective is concave. It follows that solutions to the first-order conditions will be maxima.

The Lagrangian is $\mathcal{L} = x - e^{-y} - \lambda(x + 3y - 30) + \mu_x x + \mu_y y$. The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - \lambda + \mu_x = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = e^{-y} - 3\lambda + \mu_y = 0.$$

Case I: $x, y > 0$. Here $\mu_x = \mu_y = 0$ by complementary slackness. Then $\lambda = 1$ and $e^{-y} = 3$. It follows that $y = -\ln 3 < 0$. This violates the constraint that $y \geq 0$, so there is no solution with $x, y > 0$.

Case II: $x = 0$. Then $\lambda = 1 + \mu_x \geq 1$, so the budget constraint binds. It follows that $y = 10$. Then $e^{-10} + \mu_y = 3\lambda \geq 3$. This requires $\mu_y > 0$ when $y = 0$ by complementary slackness. This contradiction means there is no solution with $x = 0$.

Case III: $x > 0$, $y = 0$. Then $\lambda = 1$. and so $\mu_y = 2$. This satisfies the first order conditions. Since $\lambda = 1$, the budget constraint binds by complementary slackness, implying $x = 30$.

Case IV: $x = 0$, $y = 0$. Then $\lambda = 0$ by complementary slackness. By the first-order conditions, $\mu_x = -1$, which is impossible. This is not a solution.

It follows that the maximum is at $(30, 0)$, where $u(30, 0) = 29$.

4. Consider the difference equation

$$\mathbf{x}_{t+1} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \mathbf{x}_t.$$

a) Find the eigenvalues of the system.

b) Find eigenvectors corresponding to the eigenvalues.

c) Do the eigenvectors form a basis for $\mathbb{R}^2$? Explain.

d) Write the general solution of this difference equation.

Answer:

a) The characteristic equation is $\lambda^2 - 4 = 0$. The eigenvalues are $\lambda = \pm 2$.

b) For $\lambda = 2$, the eigenvectors must obey

$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0}.$$

This is satisfied by any non-zero multiple of $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

For $\lambda = 02$, the eigenvectors must obey

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \mathbf{v} = \mathbf{0}.$$

This is satisfied by any non-zero multiple of $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

c) Since the eigenvalues are distinct, the eigenvectors must form a basis. Alternatively, the fact that $\det[\mathbf{v}_1, \mathbf{v}_2] = -4$, which is non-zero, shows that we have two linearly independent vectors which must form a basis for $\mathbb{R}^2$.

d) The general solution is $\mathbf{x}_t = c_1 2^t \mathbf{v}_1 + c_2(-2)^t \mathbf{v}_2$ or

$$\mathbf{x}_t = c_1 2^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2(-2)^t \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

5. Find and classify all critical points of the function $u(x,y) = x^2 + 2xy + 2y^3$ subject to the constraint $x + 3y \le 3$. Don't forget to check constraint qualification and the second-order conditions.

Answer: We rewrite the constraint $g(x,y) = 3 - x - 3y \ge 0$. Then $dg = -(1,3)$. Thus dg has maximum rank (i.e, rank one), satisifying constraint qualification. (We don't have to worry about the case where g doesn't bind.) Alternatively, the constraint is linear, so constraint qualification is satisfied.

The Lagrangian is $\mathcal{L} = x^2 + 2xy + 2y^3 + \lambda(3 - x - 3y)$. The multiplier λ will obey $\lambda \ge 0$ for a maximum and $\lambda \le 0$ for a minimum. The first-order conditions are:

$$\begin{aligned} 0 &= 2x + 2y - \lambda \\ 0 &= 2x + 6y^2 - 3\lambda. \end{aligned}$$

If the constraint does not bind (i.e., $x + 3y < 3$), $\lambda = 0$ by complementary slackness. The first equation then tells us $y = -x$ and the second yields $y = 0$ or $y = 1/3$. Both solutions satisfy the constraint yielding critical points of $(0,0)$ and $(-1/3, 1/3)$. The Hessian is then

$$\mathbf{H} = \begin{pmatrix} 2 & 2 \\ 2 & 12y \end{pmatrix}.$$

We check the leading principal minors. At $(0,0)$, $|\mathbf{H}_1| = 2 > 0$ and $|\mathbf{H}_2| = -4 < 0$, so it is neither a maximum nor minimum. At $(-1/3, 1/3)$, $|\mathbf{H}_1| = 2 > 0$ and $|\mathbf{H}_2 = 4 > 0$, so it is a local minimum.

This leave the case where the constaint binds, where $x = 3 - 3y$. Substituting in the first-order conditions we find

$$\begin{aligned} \lambda &= 6 - 4y \\ \lambda &= 2 - 2y + 2y^2. \end{aligned}$$

Eliminating λ, we find $2y^2 + 2y - 4 = 0$. This has solutions $y = 1$ and $y = -2$. The corresponding critical points are $(0,1)$ and $(9,-2)$, both with $\lambda > 0$. Here we must consider the bordered Hessian

$$\mathbf{B} = \begin{pmatrix} 0 & -1 & -3 \\ -1 & 2 & 2 \\ -3 & 2 & 12y \end{pmatrix}.$$

With two variables and one constraint, we only look at the determinant $\det \mathbf{B} = -6-12y$. This is negative when $y = 2$ (local minimum) and positive when $y = -1$ (local maximum).

I did not ask about global maxima or minima, but $\lim_{t\to\pm\infty} f(0, y) = \pm\infty$, so it has neither a global maximum nor global minimum.

In summation, there are four critical points: $(0, 0)$, $(-1/3, 1/3)$, $(0, 1)$, and $(9, -2)$. The first is a saddlepoint, the second and fourth local minima, and the third a local maximum.