Mathematical Economics Exam #1, October 5, 2017

1. Consider the sequence defined by $x_n = 1 + (-1)^n/n^2$.

   a) Does this sequence converge? If so, demonstrate it. If not, show it does not converge.

   b) Find a convergent subsequence of $y_n = (-1)^n + 1/n^2$.

   Answer:

   a) This sequence converges to 1. To see this consider $|x_n - 1| = 1/n$. Let $\varepsilon > 0$ be given. We can then choose N with $1/N^2 < \varepsilon$. For $n > N$, $1/n^2 < 1/N^2 < \varepsilon$, so $|x_n - 1| < \varepsilon$ for $n > N$, establishing that $x_n \to 1$.

   b) Although this sequence does not converge, the subsequence $y_{2n} = 1 - 1/4n^2$ converges to 1. Part (a) can be easily adapted to show this. The subsequence $y_{2n+1} = -1 + 1/(2n+1)^2$ also converges, to -1.

2. Consider the linear system

$$
\begin{aligned}
x + 2y + 3z + w &= 3 \\
2y + z + w &= 5 \\
10x + y + z + 3w &= 5
\end{aligned}
$$

   a) Does this system have a solution? If so, are there free variables?

   b) Find all solutions of the system.

   Answer: We form the augmented matrix and row reduce.

$$
\begin{pmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 2 & 1 & 1 & 5 \\ 10 & 1 & 1 & 3 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 2 & 1 & 1 & 5 \\ 0 & -19 & -29 & -7 & -25 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1/2 & 1/2 & 5/2 \\ 0 & -19 & -29 & -7 & -25 \end{pmatrix}
$$

$$
\to \begin{pmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 1/2 & 1/2 & 5/2 \\ 0 & 0 & -39/2 & 5/2 & 45/2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 1/2 & 1/2 & 5/2 \\ 0 & 0 & 1 & -5/39 & -15/13 \end{pmatrix}
$$

$$
\to \begin{pmatrix} 1 & 0 & 0 & 10/39 & 4/13 \\ 0 & 1 & 1/2 & 1/2 & 5/2 \\ 0 & 0 & 1 & -5/39 & -15/13 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 10/39 & 4/13 \\ 0 & 1 & 0 & 22/39 & 125/26 \\ 0 & 0 & 1 & -5/39 & -15/13 \end{pmatrix}
$$

   a) From this, we see that there is one free variable and the system has infinitely many solutions.

b) The solutions are given by $x = 4/13 + 10w/39$, $y = -125/26 - 22w/39$, $z = 15/13 + 5w/39$ with any $w \in \mathbb{R}$.

3. Consider the set

$$\mathscr{B} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$

a) Is $\mathscr{B}$ a basis?

b) If $\mathscr{B}$ is a basis, show that $(3,2,1)^\dagger$ is in the span of $\mathscr{B}$. If $\mathscr{B}$ is not a basis, find a non-zero vector that is **not** in the span of $\mathscr{B}$.

Answer:

a) We take the determinant of the matrix formed by the vectors in $\mathscr{B}$. Now

$$\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ 2 & 1 & 4 \end{vmatrix} = 16 - 16 = 0.$$

Since the determinant is zero, $\mathscr{B}$ is **not** a basis.

b) There are various ways to solve this. One is to find a vector that is perpendicular to the first two vectors (since $\mathscr{B}$ is not a basis, the third vector must be a linear combination of the other two, and so also perpendicular). Anything of the form $(2\alpha, \beta, -\alpha)$ is perpendicular to $(1,0,2)^\dagger$. To also be perpendicular to $(1,2,1)^\dagger$ requires $\alpha + 2\beta = 0$. We set $\alpha = 1$ and $\beta = -1/2$, yielding $(2,-1/2,-1)^\dagger$ as the desired vector. (Other solutions are possible).

4. On $\mathbb{R}^L$, let

$$\|\mathbf{x}\|_2 = \left(\sum_{\ell=1}^{L} |x_\ell|^2\right)^2$$

and

$$\|\mathbf{x}\|_\infty = \max\{|x_\ell| : \ell = 1, 2, \ldots, L\}.$$

a) Show that $\|\mathbf{x}\|_2 \leq L^{1/2}\|\mathbf{x}\|_\infty$ and $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^L$.

b) Use (a) to show that $\{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\|_\infty < 1\} \subset \{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\|_2 < L^{1/2}$.

Answer:

a) First, $\|\mathbf{x}\|_2^2 = \sum_\ell |x_\ell|^2 \leq \sum_\ell \|\mathbf{x}\|_\infty = L\|\mathbf{x}\|_\infty$, so $\|\mathbf{x}\|_2 \leq L^{1/2}\|\mathbf{x}\|_\infty$. Second, $\|\mathbf{x}\|_2^2 \geq |x_\ell|^2$ for all ℓ, so $\|\mathbf{x}\|_2 \geq \max |x_\ell| = \|\mathbf{x}\|_\infty$.

b) We use the first result from (a). If $\|\mathbf{y}-\mathbf{x}\|_\infty < 1$, then $L^{-1/2}\|\mathbf{y}-\mathbf{x}\|_2 < \|\mathbf{y}-\mathbf{x}\|_\infty < 1$, so $\|\mathbf{y}-\mathbf{x}\|_2 < L^{1/2}$. The result immediately follows.

NB: The other part of (a) can be used to show that $\{\mathbf{y} : \|\mathbf{y}-\mathbf{x}\|_2 < 1 \subset \{\mathbf{y} : \|\mathbf{y}-\mathbf{x}\|_\infty < 1$. Thus balls in one norm are nested in balls of the other, and vice-versa. Such norms are called *equivalent* and have the same open sets.