Mathematical Economics Exam #2, Nov. 9, 2017

1. Consider the problem of maximizing $p_1 x_1 + p_2 x_2$ under the constraint $x_1^4 + x_2^4 \leq 1$. The parameters obey $p_1 p_2 \neq 0$.

   a) Is constraint qualification satisfied?

   **Answer:** There is only one constraint, with derivative $dg = (4x_1^3, 4x_2^3)$. This is non-zero unless $x = 0$. As there is is one constraint, the non-degenerate constraint qualification condition (NDCQ) is satisfied for $x \neq 0$. When the constraint binds, $x \neq 0$, so NDCQ is satisfied.

   b) Find all solutions to the maximization problem.

   **Answer:** Form the Lagrangian $L = p_1 x_1 + p_2 x_2 + \lambda(1 - x_1^4 - x_2^4)$. The first order conditions are

   \[
   0 = p_1 - 4\lambda x_1^3 \\
   0 = p_2 - 4\lambda x_2^3
   \]

   It follows that $\lambda > 0$, so $x_1^4 + x_2^4 = 1$ (the constraint binds by complementary slackness). Then $p_1/p_2 = x_1^3/x_2^3$, so $x_1^4 = (p_2/p_1)^{4/3} x_1^4$. Substituting into the constraint, we find $[1 + (p_2/p_1)^{4/3}] x_1^4 = 1$. Since $\lambda > 0$, $x_i$ must have the same sign as $p_i$.

   \[
   \begin{pmatrix}
   x_1 \\
   x_2
   \end{pmatrix} = \frac{1}{(p_1^{4/3} + p_2^{4/3})^{1/4}} \begin{pmatrix}
   (\text{sgn } p_1) p_1^{1/3} \\
   (\text{sgn } p_2) p_2^{1/3}
   \end{pmatrix}.
   \]

   It follows that the maximum value of $p_1 x_1 + p_2 x_2$ is

   \[
   \left( p_1^{4/3} + p_2^{4/3} \right)^{3/4}.
   \]

   Although it is beyond the scope of this problem, we can now obtain a more general result. For $1 \leq p < \infty$, define $\|x\|_p$ by

   \[
   \|x\|_p = \left( \sum_{i=1}^{L} |x_i|^p \right)^{1/p}.
   \]

   The same technique shows that in $\mathbb{R}^L$, maximizing $p \cdot x$ subject to the constraint $\|x\|_p \leq 1$ for $1 < p < \infty$, yields maximum value $\|p\|_q$ with $1/p + 1/q = 1$. Thus
for any $x \neq 0$ $p \cdot (x/\|x\|_p) \leq \|p\|_q$, implying that $p \cdot x \leq \|p\|_q \|x\|_p$. Consideration of $-x$ allows us to write this as $|p \cdot x| \leq \|p\|_q \|x\|_p$, which is a special case of Hölder’s Inequality.

2. Consider the problem of maximizing $\sqrt{c\ell}$ under the constraints $c + \ell \leq 24$, $c + 3\ell/4 \leq 20$, $c \geq 0$, and $\ell \geq 0$.

   a) Sketch the feasible set.

   **Answer:**

   \[ \begin{array}{c}
   c + \ell = 24 \\
   c + 3\ell/4 = 20 \\
   24 \quad 26\frac{2}{3} \\
   \ell
   \end{array} \]

   b) Is constraint qualification satisfied?

   **Answer:** Two of the constraints must be rewritten as $-c \leq 0$ and $-\ell \leq 0$. Then

   \[ dg = \begin{pmatrix} 1 & 1 \\ 1 & 3/4 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \]

   Each line is non-zero, so if only one constraint binds, constraint qualification is satisfied. Part (a) makes it clear that at most two constraints can bind. Any pair of rows of $dg$ are linearly independent, so the rank is two in any of these cases. This means that the non-degenerate constraint qualification condition (NDCQ) is satisfied.

   c) Find all solutions to the maximization problem.
**Answer:** The Lagrangian is $L = \sqrt{c\ell} + \lambda_1 c + \lambda_2 \ell + \lambda_3 (24 - c - \ell) + \lambda_4 (20 - c - 3\ell/4)$. The first order conditions are

$$
0 = \sqrt{\ell/c} + \lambda_1 - \lambda_3 - \lambda_4 \\
0 = \sqrt{c/\ell} + \lambda_2 - \lambda_3 - 3\lambda_4/4.
$$

These conditions cannot be satisfied if either $c = 0$ or $\ell = 0$. By complementary slackness, $\lambda_1 = \lambda_2 = 0$. Complementary slackness also shows that at least one of constraints 3 and 4 must bind. We now rewrite the first order conditions

$$
\sqrt{\ell/c} = \lambda_3 + \lambda_4 \\
\sqrt{c/\ell} = \lambda_3 + 3\lambda_4/4.
$$

First, consider the case $c + \ell < 24$. Then $\lambda_3 = 0$. The first order conditions can be reduced to $\ell/c = 4/3$. Further, since $\lambda_4 > 0$, $c + 3\ell/4 = 20$, so $c = 10$ and $\ell = 40/3$. Then $c + \ell = 232/3 < 24$, so all the constraints are satisfied.

Next, suppose $c + 3\ell/4 < 20$. Then $\lambda_4 = 0$, and we have $\ell/c = 1$. Since $\lambda_3 > 0$, $c + \ell = 24$, implying $c = \ell = 12$. But then $c + 3\ell/4 = 21 > 21$, contradicting our assumption that $c + 3\ell/4 < 20$. There is no solution here.

The only solution is $(c, \ell) = (10, 40/3)$.

3. Consider the set $A = \{(x, y, z) : x + 2y + 3z = 6\}$.

   a) Is $A$ a vector subspace? Explain.
   b) Is $A$ open? Explain.
   c) Is $A$ connected? Explain.
   d) Is $A$ closed? Explain.
   e) Is $A$ compact? Explain.

**Answer:** The set $A$ is a plane that does not go through the origin.

   a) **No.** The point $(1, 1, 1) \in A$, but $2(1, 1, 1) \not\in A$, so it is not a vector subspace.

   b) **No.** The point $(1, 1, 1) \in A$, but $(1 + \varepsilon, 1, 1) \not\in A$ for $\varepsilon > 0$. This means any $B_\varepsilon(1, 1, 1)$ is not contained in $A$.

   c) **Yes.** Suppose $(x, y, z), (x', y', z') \in A$. Consider the path defined by $f(t) = t(x', y', z') + (1 - t)(x, y, z)$. Then $f(0) = (x, y, z), f(1) = (x', y', z')$ and $f(t) \in A$ for all $t \in [0, 1]$. This means that $A$ is path-connected, hence connected.
d) **Yes.** The set is closed because \( f(x, y, z) = x + 2y + 3z \) is continuous and \( A \) is the inverse image of a closed set, \( A = f^{-1}\{6\} \).

c) **No.** The set is not bounded because \((6, n, -n) \in A\) for all \(n\).

4. Let \( f(x, y, z) = x^2 + y^2 + 3z \).

   a) Does this function map \( \mathbb{R}^3 \) onto \( \mathbb{R} \)?

   **Answer:** Yes, it is onto. In fact, we can even set two of the variables to zero. Here \( f(0, 0, a/3) = a \), showing that the function takes all real values.

   b) Find all points \((x_0, y_0, z_0)\) satisfying \( f(x_0, y_0, z_0) = 12 \).

   **Answer:** Such points obey \( x_0^2 + y_0^2 = 12 - 3z_0 \). This only has a solution when \( 12 - 3z_0 \geq 0 \). Thus the solutions are all \((x_0, y_0, z_0)\) obeying \( x_0^2 + y_0^2 = 12 - 3z_0 \) with \( z_0 \leq 4 \).

   c) For what \((x_0, y_0, z_0)\) obeying \( f(x_0, y_0, z_0) = 12 \) is there a differentiable function \( g(y, z) \) defined on some neighborhood of \((y_0, z_0)\) that obeys \( x_0 = g(y_0, z_0) \) and \( f(g(y, z), y, z) = 12 \)?

   **Answer:** Since \( \frac{\partial f}{\partial x} = 2x \), \( \frac{\partial f}{\partial x}(x_0, y_0, z_0) = 2x_0 \). The Implicit Function Theorem yields such a function \( g \) whenever \( x_0 \neq 0 \), which requires \( z_0 < 4 \). Alternatively, note that \( g(y, z) = (\text{sgn} x_0)(12 - 3z_0 - z^2)^{1/2} \) works.

   d) Compute \( dg \) at \((x_0, y_0, z_0)\) obeying \( f(x_0, y_0, z_0) = 12 \).

   **Answer:** By the Implicit Function Theorem,

   \[
   dg = -\frac{1}{2x} \left( \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = -\left( \frac{y_0}{x_0}, \frac{3}{2x_0} \right).
   \]