Mathematical Economics Final, December 4, 2018

1. On $\mathbb{R}^2$, maximize $3x + y$ under the constraint $x^2 + y \leq 9$. Note that there are no non-negativity constraints. Don't forget to check constraint qualification. What do the second order conditions tell you? Is your solution a maximum?

   Answer: The derivative of the constraint is $(2x, 1)$, which has rank one, satisfying constraint qualification.

   The Lagrangian is $L = 3x + y - \lambda(x^2 + y - 9)$. The first order conditions are

$$0 = 3 - 2\lambda x$$
$$0 = 1 - \lambda.$$

   This implies $\lambda = 1$ and $x = 3/2$. Using the constraint, we find the solution is $(x, y) = (3/2, 27/4)$.

   That leaves the second order conditions. We form the bordered Hessian using the Lagrangian and evaluating at $(3/2, 27/4)$ to obtain

$$\mathbf{B} = \begin{pmatrix} 0 & 3 & 1 \\ 3 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

   The determinant is $2 > 0$. Since there are two variables, this must have the same sign as $(-1)^2 = +1$, which it does. The second order conditions are satisfied.

2. On $\mathbb{R}^2_+$, find demand by maximizing the utility function $u(x, y) = 3x - e^{-y}$ subject to the constraints $p_x x + p_y y \leq m$, $x \geq 0$, $y \geq 0$ where $p_x, p_y, m > 0$. Don't forget to check constraint qualification and the second order conditions.

   Answer: The derivative of the constraint equations is

$$\begin{pmatrix} p_x & p_y \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   As is usual with budget sets, at most two constraints can simultaneously bind. Since each row of the matrix is non-zero, and any pair are linearly independent, constraint qualification is satisfied.

   The Lagrangian is $L = 3x - e^{-y} - \lambda(p_x x + p_y y - m) + \mu_x x + \mu_y y$.

We note that u is concave. Since the constraints are linear, the second order conditions are automatically satisfied.

Alternatively, we can calculate the bordered Hessian of the Lagrangian.

$$\mathbf{H} = \begin{pmatrix} 0 & p_x & p_y \\ p_x & 0 & 0 \\ p_y & 0 & -e^{-y} \end{pmatrix}.$$

We need only check the determinant, which is $\det \mathbf{H} = p_x^2 e^{-y} > 0$. As there are $n = 2$ variables, the determinant must have the same sign as $(-1)^2 = 1$, which it does. The second order conditions are satisfied.

The first order conditions are

$$\begin{aligned} 0 &= 3 - \lambda p_x + \mu_x \\ 0 &= e^{-y} - \lambda p_y + \mu_y. \end{aligned}$$

These can be rewritten

$$\begin{aligned} \lambda p_x &= 3 + \mu_x \\ \lambda p_y &= e^{-y} + \mu_y. \end{aligned}$$

We first consider the case where $x, y > 0$. Then $\lambda p_x = 3$, so $\lambda > 0$. This means the budget constraint holds with equality. Moreover, $3p_y/p_x = e^{-y} \leq 1$. As a result, this case requires $p_x/p_y \geq 3$. Then $y = \ln p_x/3p_y \geq 0$ and $x = (m - p_y \ln p_x/3p_y)/p_x$. Since this is positive, we must have $m > \ln p_x/3p_y$ for this case to have a solution. Here $(x, y) = \big((m - \ln p_x/3p_y)/p_x, \ln p_x/3p_y\big)$.

Now suppose $x > 0$. Then $\lambda = 3/p_x > 0$, so the budget constraint holds with equality. By the other first order equation, $3p_y/p_x = e^{-y} + \mu_y$. We've already considered the case $y > 0$. If $y = 0$, we must have $p_x/p_y < 3$. Then $(x, y) = (m/p_x, 0)$.

Finally, there is the case $y > 0$, $x = 0$. Then $y = m/p_y$. The first order condition for y becomes $\lambda p_y = e^{-m/p_y}$. This must satisfy $\lambda p_x \geq 3$. Combining these, we find $p_y/3p_x \geq e^{m/p_y}$, or $p_y \ln p_y/3p_x \geq m$.

That leaves us with the following cases:

1. If $p_x/p_y \leq 3$, $(x, y) = (m/p_x, 0)$.
2. If $p_x/p_y > 3$ and $m \leq p_y \ln p_x/3p_y$, then $(x, y) = (0, m/p_y)$.
3. If $p_x/p_y > 3$ and $m > p_y \ln p_x/3p_y$, then $(x, y) = \big((m - p_y \ln p_x/3p_y)/p_x, \ln p_x/3p_y\big)$.

3. Consider the differential equation $\ddot{y} - 4\dot{y} + 3y = t + 1$ with initial conditions $y(0) = 7/3$ and $\dot{y}(0) = 4/3$.

   a) Find the general solution of the associated homogeneous equation.

   b) Find a particular solution of the inhomogeneous equation.

   c) Find the solution that obeys the initial conditions.

   Answer:

   a) We substitute $y = e^{rt}$ to find the characteristic equation, $r^2 - 4r + 3 = 0$. This has solutions $r = 1$ and $r = 3$. The corresponding general solution to the homogeneous equation is $c_1 e^t + c_2 c^{3t}$.

   b) We try a solution of the form $y = at + b$. Then $\dot{y} = a$ and $\ddot{y} = 0$, yielding $-4a + 3at + 3b = t + 1$, so $a = 1/3$ and $b = 7/9$. The particular solution is $t/3 + 7/9$.

   c) Combining (a) and (b), we find the general solution to the original equation is $y(t) = c_1 e^t + c_2 e^{3t} + t/3 + 7/9$. Then $y(0) = c_1 + c_2 + 7/9 = 7/3$, so $c_1 + c_2 = 14/3$. Also, $\dot{y}(0) = c_1 + 3c_2 + 1/3 = 4/3$, so $c_1 + 3c_2 = 1$. Solving for c_1, c_2, we obtain $c_1 = 11/6$ and $c_2 = -5/18$, so $y(t) = \frac{11}{6} e^t - \frac{5}{18} e^{3t} + t/3 + 7/9$.

4. Let $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$.

   a) Use the Implicit Function Theorem to show that if $\mathbf{x}^0 = (x_1^0, x_2^0, x_3^0) \in S$, we can write one of the coordinates in terms of the other two in some neighborhood of $\mathbf{x}^0$.

   b) Give an example of how to do this at the point $\mathbf{x}^0 = (0, 1, 0)$.

   Answer:

   a) Let $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 1$, so S is the set of points where $f(x_1, x_2, x_3) = 0$. Then $df = (2x_1, 2x_2, 2x_3)$. Since $(x_1^0)^2 + (x_2^0)^2 + (x_3^0)^2 = 1$, there is some coordinate $x_i^0 \neq 0$ with $\partial f/\partial x_i \neq 0$. This allows us to apply the Implicit Function Theorem.

   b) Here $\partial f/\partial x_2 \neq 0$, so we can write x_2 as a function of (x_1, x_3). In fact, $x_2(x_1, x_3) = +\sqrt{1 - x_1^2 - x_3^2}$ will do.

5. Consider the difference equation $\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t$ where

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 3/2 \end{pmatrix}.$$

   a) Find the eigenvalues.

   b) Find a non-zero vector $\mathbf{x}$, so that if $\mathbf{x}_0 = \mathbf{x}$, then $\mathbf{x}_t \to \mathbf{0}$.

Answer:

a) The characteristic equation is $0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - (3/2)\lambda - 1$. This has solutions $\lambda = 2$ and $\lambda = -1/2$.

b) The stable root is $\lambda = -1/2$, so we find an eigenvector for it. We solve

$$(\mathbf{A} + \frac{1}{2}\mathbf{I})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then $(x_1, x_2) = (2, 1)$ or any non-zero multiple thereof. If the initial conditions are $\mathbf{x}_0 = (2, 1)^{\mathbf{T}}$, the solution is $\mathbf{x}_t = (1/2)^t(2, 1)^{\mathbf{T}} \to \mathbf{0}$.