Mathematical Economics Exam #1, October 1, 2019

1. Consider the set

$$\mathscr{B} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

   a) Is $\mathscr{B}$ a basis?

   b) If $\mathscr{B}$ is a basis, find the coordinates of the vector $\mathbf{x} = (1,2,3)^{\mathsf{T}}$ in the basis $\mathscr{B}$. If $\mathscr{B}$ is not a basis, find a non-zero vector that is **not** in the span of $\mathscr{B}$.

Answer:

   a) We take the determinant of the matrix formed by the vectors in $\mathscr{B}$. Now

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 3 - 0 = 3.$$

   Since the determinant is non-zero, $\mathscr{B}$ **is** a basis.

   b) To find the coordinates, we let $\mathbf{B}$ be the matrix formed by the basis $\mathscr{B}$. Then $\mathbf{B}^{-1}\mathbf{x}$ will be the coordinates.

   Row reduction yields the inverse

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1/3 & 1 & -1/3 \\ -1/3 & 0 & 1/3 \end{pmatrix}$$

   The coordinates are then

$$\mathbf{x}_{\mathscr{B}} = \begin{pmatrix} -1 \\ 4/3 \\ 2/3 \end{pmatrix}$$

   meaning that

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{4}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}.$$

2. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

   a) Compute the rank of $\mathbf{A}$.

   b) What is the dimension of the nullspace (kernel) of $\mathbf{A}$, $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{0}\}$?

   c) Find a basis for the nullspace of $\mathbf{A}$.

   Answer:

   a) This matrix is alrealy in row echelon form. There are two non-zero rows, so the rank is 2.

   b) If $\mathbf{A}\mathbf{x} = \mathbf{0}$, we have $x_2 = 0$ and $x_3 = 0$. There are two bound variables and one free variable (x_1). It follows that the kernel has dimension 1.

   c) It follows from (b) that $(1, 0, 0)^\mathsf{T}$ is a basis for the nullspace.

3. Consider the following sets:

$$S = \cup_{n=1}^{\infty}(-n, 0), \qquad T = \cap_{n=1}^{\infty}(-n, +n).$$

   a) Is S an open set? Explain

   b) Is T open? Closed? Explain.

   Answer:

   a) As the union of open intervals, $S = (-\infty, 0)$ is open.

   b) Since T is an **infinite** intersection of open sets, we cannot use general principles to determine whether it is open. We must compute T. Here $T = (-1, +1)$, which is an open interval. It is not a closed set because the limit points $+1$ and -1 are not in T.

4. Consider the sequence

$$x_n = \left\{ (-1)^n \frac{n+1}{n+3} \right\}_{n=1}^{\infty}.$$

   Does this sequence converge? If so, find the limit. If not, does it have any cluster points. Explain.

   Answer: The quotient converges to 1, so the sequence eventually alternates between being very close to -1 and $+1$. This means that both -1 and $+1$ are cluster points. With two distinct cluster points, it cannot have a limit, since having a limit would mean that the tail of the sequence is eventually within any ε of the limit. But it is within ε of both $+1$ and -1 infinitely often, so the limit must be within ε of both, which is impossible for $\varepsilon < 1$ since ± 1 are 2 units apart.