Mathematical Economics Final, December 10, 2020

1. On $\mathbb{R}^2$, maximize $3x + 5y$ under the constraints $2x + y \leq m$, $x \geq 0$ and $y \geq 0$ where $m > 0$. Don't forget to check constraint qualification. Interpret the multiplier for the constraint $2x + y \leq m$.

   Answer: The constraints have derivative

$$Dg = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

At most two constraints can simultaneously bind since $m > 0$. The rank of any two rows of Dg is 2, and the rank of any single row is 1. It follows that NDCQ is satisfied at all feasible points.

We start by forming the Lagrangian

$$\mathcal{L} = 3x + 5y - \lambda(2x + y - m) + \mu_x x + \mu_y y.$$

The first order conditions are

$$\begin{aligned} 0 &= 3 - 2\lambda + \mu_x \\ 0 &= 5 - \lambda + \mu_y. \end{aligned}$$

or

$$\begin{aligned} 2\lambda &= 3 + \mu_x \\ \lambda &= 5 + \mu_y. \end{aligned}$$

By non-negatively of the multipliers, $\lambda \geq 3 > 0$. Then complementary slackness implies $2x + y = m$.

We have three cases to consider: (1) $x, y > 0$, (2) $x > 0$ and $y = 0$, and (3) $x = 0$, $y > 0$.

Case 1: This case fails. By complementary slackness, $\mu_x = \mu_y = 0$, so $\lambda = 3/2$ and $\lambda = 5$, which is impossible.

Case 2: Here $\mu_x = 0$, so $\lambda = 3/2$. It follows that $\mu_y = 7/2$, which is ok. Then $y = 0$ and $x = m/2$ by the budget constraint and utility (the objective) is $3m/2$.

Case 3: Here $\mu_y = 0$, so $\lambda = 5$. It follows that $\mu_x = 7$ which is ok since $x = 0$ in this case. Also, $y = m$. This yields utility $5m$.

Now case 3 is the maximum, so $x = 0$, $y = m$ and the maximum value is $5m$.

As is typical is consumer's problems, the multiplier is the marginal (indirect) utility of income by the Envelope Theorem. Here indirect utility is $v(m) = 5m$, so $dv/dm = 5 = \lambda$.

2. Let $u(x_1, x_2) = x_1^2 + x_2^2$. Solve the expenditure minimization problem for $\bar{u} > 0$ and $\mathbf{p} \gg \mathbf{0}$. Compute both the Hicksian demands and the expenditure function e.

$$\begin{aligned} e(\mathbf{p}, \bar{u}) = \min\ & \mathbf{p}\cdot\mathbf{x} \\ \text{s.t. } & u(\mathbf{x}) \geq \bar{u} \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Answer: The constraints have derivative

$$D\mathbf{g} = \begin{pmatrix} 2x_1 & 2x_2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

At most two constraints can simultaneously bind (see the diagram). Since $\bar{u} > 0$, $\mathbf{x} \neq \mathbf{0}$. It follows that the rank of any two rows of $D\mathbf{g}$ is 2, and the rank of any single row is 1. It follows that NDCQ is satisfied at all feasible points.

We form the Lagrangian

$$\mathcal{L} = p_1x_1 + p_2x_2 - \lambda(x_1^2 + x_2^2 - \bar{u}) - \mu_1x_1 - \mu_2x_2$$

The first order conditions are

$$\begin{aligned} 0 &= p_1 - 2\lambda x_1 - \mu_1 \\ 0 &= p_2 - 2\lambda x_2 - \mu_2. \end{aligned}$$

or

$$\begin{aligned} p_1 &= 2\lambda x_1 + \mu_1 \\ p_2 &= 2\lambda x_2 + \mu_2. \end{aligned}$$

There are three cases: (1) $x_1, x_2 > 0$, (2) $x_1 > 0$ and $x_2 = 0$, and (3) $x_1 = 0$, $x_2 > 0$.

Case 1: Here $\mu_1 = \mu_2 = 0$ by complementary slackness. Then $p_i = 2\lambda x_i$. This requires $\lambda > 0$, so $x_1^2 + x_2^2 = \bar{u}$ by complementary slackness. Also, $x_i = p_i/2\lambda$, so

$$\bar{u} = x_1^2 + x_2^2 = (p_1^2 + p_2^2)/4\lambda^2.$$

It follows that $2\lambda = \bar{u}^{-1/2}(p_1^2 + p_2^2)^{1/2}$. The demands are

$$x_i = \frac{p_i}{2\lambda} = p_i \bar{u}^{1/2} \frac{1}{(p_1^2 + p_2^2)^{1/2}}$$

yielding expenditure $e(\mathbf{p}, \bar{u}) = \bar{u}^{1/2} \left(p_1^2 + p_2^2\right)^{1/2}$.

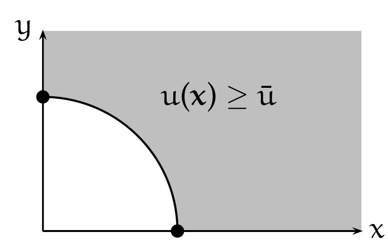
Case 2: Here $x_1 > 0$ and $x_2 = 0$. Then $p_1 = 2\lambda x_1$, so $\lambda > 0$. By complementary slackness, $\bar{u} = x_1^2 + x_2^2 = x_1^2$ so $x_1 = \sqrt{\bar{u}}$ (the negative square root is forbidden by the constraints). Here $e(\mathbf{p}, \bar{u}) = p_1\sqrt{\bar{u}}$.

Case 3 is similar, with $x_1 = 0$ and $x_2 = \sqrt{\bar{u}}$. Finally $e(\mathbf{p}, \bar{u}) = p_2\sqrt{\bar{u}}$.

Of the three solutions, case 1 has higher expenditure than either case 2 or 3. It is not the minimum. In fact, it yields the maximum expenditure among points in the positive orthant with $x_1^2 + x_2^2 = \bar{u}$. Of the other two, the cheaper option is to spend everything on the cheapest good. It follows that the true expenditure function is

$$e(\mathbf{p}, \bar{u}) = \bar{u}^{1/2} \min\{p_1, p_2\}.$$

As for the Hicksian demand, let $k = \operatorname{argmin}_{i=1,2} p_i$ and let ℓ be the other good. Then $x_\ell = 0$ and $x_k = \sqrt{\bar{u}}$. If $p_1 = p_2$, either $(\bar{u}^{1/2}, 0)$ or $(0, \bar{u}^{1/2})$ will minimize expenditure.



Constraint Set: We draw the constraint set to clarify what is happening here. The marked points are the two possible minima. The cheaper of the two will be chosen. Notice that convex combinations of these points are not feasible, and that the point where the slope of the constraint is p_1/p_2 is more costly than either of the two corner points.

3. Consider the differential equation $2\ddot{y} - 3\dot{y} + y = te^t$ with initial conditions $y(0) = 0$ and $\dot{y}(0) = -1$.

   a) Find the general solution of the associated homogeneous equation.

   b) Find a particular solution of the inhomogeneous equation.

   c) Find the solution that obeys the initial conditions.

Answer:

a) The characteristic polynomial is $2\lambda^2 - 3\lambda + 1 = 0$, which has solutions $\lambda = 1, 1/2$. The general solution to the homogeneous equation is $y_g(t) = c_1 e^t + c^2 e^{t/2}$.

b) We look for particular solutions of the form

$$y = Ate^t + Bt^2e^t$$

when

$$\begin{aligned}\dot{y} &= Ae^t + Ate^t + 2Bte^t + Bt^2e^t \\ &= Ae^t + (A + 2B)te^t + Bt^2e^t \\ \ddot{y} &= 2(A + B)e^t + (A + 4B)te^t + Bt^2e^t \\ &= te^t.\end{aligned}$$

Then

$$\begin{aligned}2\ddot{y} - 3\dot{y} + y = 4(A + B)e^t &+ 2(A + 4B)te^t + 2Bt^2e^t \\ &- 3Ae^t - 3(A + 2B)te^t - 3Bt^2e^t \\ &+ Ate^t + Bt^2e^t\end{aligned}$$

It follows that

$$\begin{aligned}0 &= 4(A + B) - 3A + A = A + 4B \\ 1 &= 2(A + 4B) - 3(A + 2B) + A = 2B \\ 0 &= 2B - 3B + B = 0.\end{aligned}$$

So $A = -2$ and $B = 1/2$. Our particular solution is $y_p = -2te^t + \frac{1}{2}t^2e^t$.

c) Now $y(t) = y_g(t) + y_p(t)$, so $y(0) = c_1 + c_2 = 0$, so $c_1 + c_2 = 0$. Also, $\dot{y} = c_1e^t + \frac{1}{2}c_2e^{t/2} - 2e^t - te^t + \frac{1}{2}t^2e^t$, so $\dot{y}(0) = c_1 + c_2/2 - 2 = -1$. It follows that $c_1 = 2$ and

$c_2 = -2$, yielding solution

$$y(t) = 2e^t - 2e^{t/2} - 2te^t + \frac{1}{2}t^2e^t.$$

4. Let $f(x, y) = (xy)^{12}$.

   a) Either show that f is quasiconcave on $\mathbb{R}^2_{++}$ or show that it is not quasiconcave.

   b) Is f homogeneous? If so, what is its the degree of homogeneity. If not, show that it is not homogeneous.

Answer:

a) **Method 1:** Now

$$f(x, y) = \left(x^{1/3}y^{1/3}\right)^{36}.$$

That means that f is a monotonic transformation of $g(x, y) = x^{1/3}y^{1/3}$. The Hessian of g is

$$D^2g = \begin{pmatrix} -\frac{2}{9}x^{-5/3}y^{1/3} & \frac{1}{9}x^{-2/3}y^{-2/3} \\ \frac{1}{9}x^{-2/3}y^{-2/3} & -\frac{2}{9}x^{1/3}y^{-5/3} \end{pmatrix}.$$

The diagonal terms are negative and $\det D^2g = \frac{1}{81}(4-1)x^{-4/3}y^{-4/3} > 0$, so g is concave by the second derivative test. As a monotonic transformation of a concave function, f is quasiconcave.

Method 2: The Hessian is

$$\begin{pmatrix} 132x^{10}y^{12} & 144x^{11}y^{11} \\ 144x^{11}y^{11} & 132x^{12}y^{10} \end{pmatrix}$$

We use $Df = (12x^{11}y^{12}, 12x^{12}y^{11})$ to form the bordered Hessian

$$\mathbf{B} = \begin{pmatrix} 0 & 12x^{11}y^{12} & 12x^{12}y^{11} \\ 12x^{11}y^{12} & 132x^{10}y^{12} & 144x^{11}y^{11} \\ 12x^{12}y^{11} & 144x^{11}y^{11} & 132x^{12}y^{10} \end{pmatrix}$$
$$= 12x^{10}y^{10}\begin{pmatrix} 0 & xy^2 & x^2y \\ xy^2 & 11y^2 & 12xy \\ x^2y & 12xy & 11x^2 \end{pmatrix}$$

The determinant is $\det \mathbf{B} = (12x^{10}y^{10})^3(2x^4y^4) > 0$. Since $(-1)^3 \det \mathbf{B} > 0$, the function is quasiconcave (Theorem 21.23.1 in the notes).

b) For $t > 0$, $f(tx, ty) = (txty)^{12} = t^{24}(xy)^{12} = t^{24}f(x, y)$, showing that f is homogeneous of degree 24.

5. Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Find $\mathbf{A}^{1/2}$

Answer: We first find the eigenvalues. The characteristic equation is $0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 4\lambda + 4 - 1 = \lambda^2 - 4\lambda + 3$. This has roots $\lambda = 1, 3$.

The corresponding eigenvectors are

$$\boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

I've normalized the eigenvectors so that we have an orthonormal basis.

The basis matrix is

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Because the basis orthonormal, its transpose is its inverse. Thus

$$\mathbf{P}^{-1} = \mathbf{P}^{\mathsf{T}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

We can then write

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}, \quad \text{where} \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{A}^{1/2} &= \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^{-1} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 + \sqrt{3} & -1 + \sqrt{3} \\ -1 + \sqrt{3} & 1 + \sqrt{3} \end{pmatrix} \end{aligned}$$