Homework Assignment #1

8.4 If you choose four numbers at random for the entries of a 2×2 matrix A, and four others for another 2×2 matrix B, AB will probably not equal BA. Carry out this proceedure a few times.

Answer: I will only give you one example. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 8 & 5 \\ 20 & 13 \end{pmatrix} \neq BA = \begin{pmatrix} 13 & 20 \\ 5 & 8 \end{pmatrix}.$$

8.24

a) Use Theorem 8.8 to prove that a 2×2 lower- or upper- triangular matrix is invertible if and only if each diagonal entry is nonzero.

b) Show that the inverse of a 2×2 lower triangular matrix is lower triangular.

c) Show that the inverse of a 2×2 upper triangular matrix is upper triangular.

Answer: Theorem 8.8 tells us that a 2×2 matrix in the form (5) is invertible if and only if $ad - bc \neq 0$, in which case the inverse is (7). For part (a), note that an upper triangular matrix has $c = 0$ while a lower triangular matrix has $b = 0$. Either way, $bc = 0$, so $ad - bc = ad$. By Theorem 8.8, such a matrix is invertible if and only if $ad \neq 0$, which is equivalent to $a \neq 0$ and $d \neq 0$, equivalent to the diagonal being non-zero.

For parts (b) and (c), note that the inverse of a (upper or lower) triangular matrix is

$$A^{-1} = \frac{1}{ad} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

When $b = 0$ (lower triangular), A^{-1} is also lower triangular by the inverse formula and when $c = 0$ (upper triangular), A^{-1} is also upper triangular by the inverse formula.

8.28 What is the inverse of the $n \times n$ diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{pmatrix}?$$

Answer: Provided $d_i \neq 0$ for $i = 1, \dots, n$, the inverse is

$$D^{-1} = \begin{pmatrix} 1/d_1 & 0 & 0 & \dots & 0 \\ 0 & 1/d_2 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1/d_n \end{pmatrix}.$$

If any $d_i = 0$, there is no inverse.

9.11 Use Theorem 9.4 to invert the following matrices:

$$a)\ \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}, \qquad c)\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Answer: In all three cases we use the formula $A^{-1} = (1/\det A) \operatorname{adj} A$.

a) The determinant is $4 - 3 = 1$, so the inverse is the adjoint, $\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$.

b) The determinant is 37. After computing the cofactors, we find that the inverse is

$$\frac{1}{37} \begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -6 \\ -5 & 2 & 5 \end{pmatrix}.$$

c) The determinant is $ad - bc$, and the inverse is $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

9.13 Use Cramer's Rule to solve the following systems of equations:

$$a)\ \begin{aligned} 5x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 4 \end{aligned} \qquad b)\ \begin{aligned} 2x_1 - 3x_2 &= 2 \\ 4x_1 - 6x_2 + x_3 &= 7 \\ x_1 + 10x_2 &= 1. \end{aligned}$$

Answer: Cramer's Rule tells us that the solution to system (a) is

$$x_1 = \begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix} \Big/ \begin{vmatrix} 5 & 1 \\ 2 & -1 \end{vmatrix} = \frac{-7}{-7} = 1$$

$$x_2 = \begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} \Big/ \begin{vmatrix} 5 & 1 \\ 2 & -1 \end{vmatrix} = \frac{14}{-7} = -2.$$

The solution to system (b) is

$$x_1 = \begin{vmatrix} 2 & -3 & 0 \\ 7 & -6 & 1 \\ 1 & 10 & 0 \end{vmatrix} \Big/ \begin{vmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0 \end{vmatrix} = \frac{-23}{-23} = 1$$

$$x_2 = \begin{vmatrix} 2 & 2 & 0 \\ 4 & 7 & 1 \\ 1 & 1 & 0 \end{vmatrix} \Big/ \begin{vmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0 \end{vmatrix} = \frac{0}{-23} = 0$$

$$x_3 = \begin{vmatrix} 2 & -3 & 2 \\ 4 & -6 & 7 \\ 1 & 10 & 1 \end{vmatrix} \Big/ \begin{vmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0 \end{vmatrix} = \frac{-69}{-23} = 3.$$

This can be easily checked by plugging the solutions back in the original equations.

26.16 Prove the following results for $n \times n$ matrices:

a) $\det r\mathbf{A} = r^n \det \mathbf{A}$

b) $\det(-\mathbf{A}) = (-1)^n \det \mathbf{A}$

c) $\det(\mathbf{A}_1 \cdots \mathbf{A}_r) = (\det \mathbf{A}_1) \cdots (\det \mathbf{A}_r)$

d) $\det \mathbf{A}^k = (\det \mathbf{A})^k$ for positive integers k

e) $\det \mathbf{A}^k = (\det \mathbf{A})^k$ for all integers k if $\mathbf{A}$ is invertible.

Answer:

a) We use that fact that det is multilinear in columns. Write $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]$ where the $\mathbf{a}_j$ are the column of $\mathbf{A}$ and write $\det \mathbf{A} = f(\mathbf{a}_1, \dots, \mathbf{a}_n)$ with f multilinear. Then $f(r\mathbf{a}_1, \dots, r\mathbf{a}_n) = r^n f(\mathbf{a}_1, \dots, \mathbf{a}_n)$ so $\det(r\mathbf{A}) = r^n \det \mathbf{A}$,

b) Set $r = -1$ in part (a), then $\det(-\mathbf{A}) = (-1)^n \det \mathbf{A}$.

c) We know $\det \mathbf{AB} = (\det \mathbf{A})(\det \mathbf{B})$. We prove the result by induction on r. Suppose it is true for r, then $\det \mathbf{A}_1 \cdots \mathbf{A}_{r+1} = (\det \mathbf{A}_1 \cdots \mathbf{A}_n)(\det \mathbf{A}_{r+1})$ by the AB case (which is also $r = 2$). Then use $\det(\mathbf{A}_1 \cdots \mathbf{A}_r) = (\det \mathbf{A}_1) \cdots (\det \mathbf{A}_r)$ to obtain $\det(\mathbf{A}_1 \cdots \mathbf{A}_{r+1}) = (\det \mathbf{A}_1) \cdots (\det \mathbf{A}_{r+1})$. That proves the induction step. Since it is true for $r = 2$, it is true for all integers $r = 2, 3, \dots$.

d) Apply (c) to $\mathbf{A}_i = \mathbf{A}$ for $i = 1, \dots, k$ to obtain the result.

e) Apply (c) to $\mathbf{A}_i = \mathbf{A}^{-1}$ for $i = 1, \dots, k$ and combine with (d) to obtain the result. Note that $k = 0$ is always the identity matrix.