Homework Assignment #8

23.15 Use Theorem 23.9 to find all the eigenvalues of the following matrices by inspection. Note that the first matrix is a Markov matrix.

$$a) \begin{pmatrix} 0.5 & 0.7 \\ 0.5 & 0.3 \end{pmatrix}, \quad b) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad c) \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad d) \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}.$$

Answer: By Theorem 23.9, the product of the eigenvalues is the determinant and the sum of the eigenvalues is the trace.

a) Here $\mathrm{tr}(\mathbf{A}) = 0.8$ and $\det \mathbf{A} = -0.2$. Because this a Markovian matrix (**every** column sums to 1), $\lambda_1 = 1$ is a eigenvalue. Then either the trace or determinant yields $\lambda_2 = -0.2$ as the other eigenvalue.

b) The repeated rows indicate this is a singular matrix, so $\lambda_1 = 0$ is an eigenvalue. Now $\mathrm{tr}(\mathbf{A}) = 2$, so $\lambda_2 = 2$ is the other eigenvalue.

c) Here $\mathrm{tr}(\mathbf{A}) = 0$ and $\det \mathbf{A} = 2$. It is also clear that $\lambda_1 = -1$ is an eigenvalue since the rows of $\mathbf{A} + \mathbf{I}$ are all the same. It follows that the other eigenvalues are $\lambda_2 = -1$ and $\lambda_3 = 2$.

d) This matrix has 2 obvious eigenvalues: $\lambda_1 = 3$ and $\lambda_2 = 0$ (because of the repeated row). Since $\mathrm{tr}(\mathbf{A}) = 6$, the third eigenvalue is also 3.

23.21

a) Show that the matrix $\mathbf{B} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ has 2 as an eigenvalue of multiplicity three, but only one independent eigenvector $\mathbf{v}_1$.

b) Use (31) to find generalized eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$.

c) If $\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$, show that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P}$ has the desired form.

Answer:

a) The eigenvalues obey $(1-\lambda)(2-\lambda)(3-\lambda) + 2 - \lambda = 0$. This can be rewritten as $(2-\lambda)^3 = 0$, showing that 2 is an eigenvalue of multiplicity 3. Then $\mathbf{B} - 2\mathbf{I} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$. If $(\mathbf{B} - 2\mathbf{I})\mathbf{v} = 0$, the components of $\mathbf{v}$ obey $v_1 + v_3 = 0$ and $v_1 + v_2 + v_3 = 0$. This means that $\mathbf{v}_1$ is a multiple of $(1, 0, -1)^T$. We set $\mathbf{v}_1 = (1, 0, -1)^T$.

b) We next find a solution to $(\mathbf{B} - 2\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$. Here $\mathbf{v}_2 = (1, 1, -1)^{\mathbf{T}}$ will do. Then set $(\mathbf{B} - 2\mathbf{I})\mathbf{v}_3 = \mathbf{v}_2$. The vector $\mathbf{v}_3 = (1, 0, 0)^T$ solves the equation.

c) Then

$$\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

This has inverse

$$\mathbf{P}^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

It is then easily verified that

$$\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

24.13 Solve the following initial value problems:

a) $\ddot{y} + 6\dot{y} + 9y = 0$, $y(0) = 0$, $\dot{y}(0) = 1$;

b) $4\ddot{y} + 4\dot{y} + y = 0$, $y(0) = 1$, $\dot{y}(0) = 1$.

Answer:

a) The characteristic equation is $r^2 + 6r + 9 = 0$, with roots -3 and -3. Due to the repeated root, the general solution is $c_1 e^{-3t} + c_2 t e^{-3t}$. Then $y(0) = c_1 = 0$, implying $c_1 = 0$. It follows that $\dot{y}(0) = c_2 = 1$, so $c_2 = 1$. The solution is $y(t) = te^{-3t}$.

b) The characteristic equation is $4r^2 + 4r + 1 = 0$, with repeated root $-1/2$. The general solution is $c_1 e^{-t/2} + c_2 t e^{-t/2}$. Then $y(0) = c_1 = 1$ and $\dot{y}(0) = -c_1/2 + c_2 = 1$. It follows that $c_1 = 1$ and $c_2 = 3/2$. The solution is $y(t) = e^{t/2} + \frac{3}{2} t e^{t/2}$.

24.17 Compute the general solution of the third order differential equation $\dddot{y} - 2\ddot{y} - \dot{y} + 2y = 0$.

Answer: We substitute $y(t) = e^{rt}$ to derive the characteristic polynomial. Then $r^3 e^{rt} - 2r^2 e^{rt} - re^{rt} + 2e^{rt} = 0$, so $r^3 - 2r^2 - r + 2 = 0$. It is easy to see that $r = 1$ is a root, so we factor $r^3 - 2r^2 - r + 2 = (r-1)(r^2 - r - 2)$. This can be further factored to $(r-1)(r-2)(r+1)$. The roots are $\{-1, +1, +2\}$, so the general solution is $y(t) = c_1 e^{-t} + c_2 e^{t} + c_3 e^{2t}$.