Mathematical Economics Exam #1, September 24, 2020

1. Consider the vectors
$$\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \text{ and } \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

   a) Do these vectors form a basis? Explain.

   b) If the vectors form a basis, find the corresponding dual basis. If the vectors do not form a basis, replace one of the vectors to form a basis, then find the dual basis of that.

   Answer:

   a) We form the basis matrix
$$\begin{pmatrix}1&0&0\\1&1&1\\0&1&1\end{pmatrix}.$$
   Since two of the columns are repeated, its determinant is zero. The vectors do not form a basis.

   b) There are many ways to do this. One is to replace the second column with $\begin{pmatrix}0&1&0\end{pmatrix}^{\mathsf{T}}$, yielding a new basis matrix
$$\mathbf{B} = \begin{pmatrix}1&0&0\\1&1&1\\0&0&1\end{pmatrix}.$$
   We invert the basis matrix. Its inverse is
$$\mathbf{B}^{-1} = \begin{pmatrix}1&0&0\\-1&1&-1\\0&0&1\end{pmatrix}.$$
   The rows of $\mathbf{B}^{-1}$ are the dual matrix, so $\mathbf{b}_1^* = (1,0,0), \mathbf{b}_2^* = (-1,1,-1)$, and $\mathbf{b}_3^* = (0,0,1)$ are the corresponding dual basis.

2. Demand for good 1 is $d_1 - ap_1 - \frac{1}{2}bp_2$; demand for good 2 is $d_2 - \frac{1}{2}ap_1 - bp_2$; the supply of good i is s_i. Here a, b, d_i, and s_i are all positive, and $s_i < d_i$. Both goods are complements.

   a) What system of equations do you get when you set supply equal to demand in both markets?

b) What criterion must be met in order to solve for p_1 and p_2? Is it satisfied?

c) What additional conditions, if any, must be satisfied in order to get positive equilibrium prices p_i?

Answer:

a)

$$\begin{aligned} d_1 - ap_1 + -\frac{1}{2}bp_2 &= s_1 \\ d_2 - \frac{1}{2}ap_1 - bp_2 &= s_2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} -ap_1 - \frac{1}{2}bp_2 &= s_1 - d_1 \\ -\frac{1}{2}ap_1 - bp_2 &= s_2 - d_2 \end{aligned}$$

b) The determinant

$$\begin{vmatrix} -a & -\frac{1}{2}b \\ -\frac{1}{2}a & -b \end{vmatrix} = ab - \frac{1}{4}ab = \frac{3}{4}ab \neq 0.$$

This is satisfied since a and b are both positive.

c) Equilibrium prices are always positive. Using Cramer's rule, we find

$$p_1 = \frac{4}{3a}\left[(d_1 - s_1) + \frac{1}{2}(d_2 - s_2)\right]$$

and

$$p_2 = \frac{4}{3b}\left[\frac{1}{2}(d_1 - s_1) + (d_2 - s_2)\right].$$

Given the conditions on the parameters, prices are always positive.

3. Let $\mathbf{B}$ be an $n \times n$ invertible matrix and define $\mathbf{A} = \mathbf{B}^\mathsf{T}\mathbf{B}$. Define $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{y}$. Is $\langle \cdot, \cdot \rangle$ an inner product on $\mathbb{R}^n$?

Answer: To check whether it is an real inner product (we're in $\mathbb{R}^n$, not $\mathbb{C}^n$), we need to see if it is symmetric, linear in the second argument, and is positive definite.

Yes. It is an inner product. First, it is symmetric. The matrix $\mathbf{A}$ is symmetric because $\mathbf{A}^\mathsf{T} = (\mathbf{B}^\mathsf{T}\mathbf{B})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{B} = \mathbf{A}$. Now considere $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{y}$. This is a real number, so it is its own transpose, meaning that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{x} = \mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x} = \langle \mathbf{y}, \mathbf{x} \rangle.$$

That proves symmetry.

Second, it is linear in the second argument because

$$\begin{aligned}\langle x, \alpha y + z\rangle &= x^{\mathsf{T}}\mathbf{A}(\alpha y + z)\\ &= x^{\mathsf{T}}\mathbf{A}(\alpha y) + x^{\mathsf{T}}\mathbf{A}z\\ &= \alpha(x^{\mathsf{T}}\mathbf{A}y) + x^{\mathsf{T}}\mathbf{A}z\\ &= \alpha\langle x, y\rangle + \langle x, z\rangle\end{aligned}$$

Finally, it is positive definite. Here

$$\langle x, x\rangle = x^{\mathsf{T}}\mathbf{A}x = x^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\mathbf{B}x = \mathbf{B}x\cdot\mathbf{B}x = \|\mathbf{B}x\|_2.$$

This is non-negative, and zero if and only if $\mathbf{B}x = 0$. Now $\mathbf{B}$ is invertible, so $\mathbf{B}x = 0$ if and only if $x = 0$.

4. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5\\ 1 & 4 & 9 & 16 & 25\\ 1 & 8 & 27 & 64 & 125\end{pmatrix}.$$

a) Find $\operatorname{rank}\mathbf{A}$.

b) Find a basis for $\ker\mathbf{A} = \{x : \mathbf{A}x = 0\}$.

c) Verify that $\operatorname{rank}\mathbf{A} + \dim\ker\mathbf{A} = 5$.

Answer:

a) We row-reduce $\mathbf{A}$ to row echelon form (no need for reduced), obtaining

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5\\ 0 & 1 & 3 & 6 & 10\\ 0 & 0 & 1 & 4 & 10\end{pmatrix}.$$

As there are three non-zero rows (and three basic variables), $\operatorname{rank}\mathbf{A} = 3$.

b) Here variables x_4 and x_5 are free and the equations for being in the kernel are

$$\begin{aligned}0 &= x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5\\ 0 &= x_2 + 3x_3 + 6x_4 + 10x_5\\ 0 &= x_3 + 4x_4 + 10x_5.\end{aligned}$$

With two free variables, any basis for $\ker \mathbf{A}$ will have two elements. We obtain a basis for $\ker \mathbf{A}$ by first setting $x_4 = 1$ and $x_5 = 0$ and solving; then setting $x_4 = 0$ and $x_5 = 1$ and solving.

This yields

$$\begin{pmatrix} -4 \\ 6 \\ -4 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -15 \\ 20 \\ -10 \\ 0 \\ 1 \end{pmatrix}.$$

$c)$ Finally, $\operatorname{rank} \mathbf{A} = 3$ and $\dim \ker \mathbf{A} = 2$, so the sum is 5, the number of variables.