Mathematical Economics Exam #2, October 28, 2020

1. Consider the equation $x^3 + y^2 = 0$.

   a) At which points is y an implicit function of x?

   b) What is $y'(x)$ at the points in (a)?

   c) Is there any point on the curve $x^3 + y^2 = 0$ where the Implicit Function Theorem fails to apply?

   Answer:

   a) Since $y^2 \geq 0$, we must have $x^3 \leq 0$, so $x \leq 0$. We can define $y(x) = +\sqrt{-x^3}$ for $x < 0$, or $y(x) = -\sqrt{-x^3}$ for $x < 0$.

   Alternatively, let $f(x,y) = x^3 + y^2$. Then $\partial f/\partial y = 2y \neq 0$ as long as $y \neq 0$. The Implicit Function Theorem shows we can define $y(x)$ in a neighborhood of any point (x_0, y_0) with $x_0^3 + y_0^2 = 0$ and $y_0 \neq 0$.

   b) When $y(x) = +\sqrt{-x^3}$, $y'(x) = -3x^2/2\sqrt{-x^3}$. When $y(x) = -\sqrt{-x^3}$, $y'(x) = +3x^2/2\sqrt{-x^3}$.

   This can also be answered using the Implicit Function Theorem if we have used it to find $y(x)$ in a neighborhood of (x_0, y_0) with $y_0 \neq 0$. Then $\partial f/\partial x = 3x^2$, so $y'(x_0) = -3x_0^2/2y_0$, which agrees with the previous answer.

   c) The Implicit Function Theorem fails to apply when $\partial f/\partial y = 2y = 0$. This happens at $y = 0$. It is impossible to define a function $y(x)$ in a neighborhood of $(0,0)$ with $[y(x)]^2 + x^3 = 0$.

2. Consider the constant elasticity demand function $Q = 12p_1^{-1/2}p_2^{3/2}$ where Q is the demand for good 1 and p_i is the price of good $i = 1,2$. Suppose current prices are $p_1 = 36$ and $p_2 = 9$.

   a) What is the current demand for Q?

   b) Use differentials to estimate the change in demand as p_1 increases by 4 and p_2 decreases by 0.5.

   c) Similarly, estimate the change in demand when both prices increase by 0.4.

   d) Estimate the totals demands for situations (b) and (c) and compare your estimates with the actual demands.

   Answer:

   a) $Q = 12(36)^{-1/2}(9)^{3/2} = 54$.

b) Now $\partial Q/\partial p_1 = -6p_1^{-3/2}p_2 3/2$. With the given values of p_i, that is $-6(6){-}33^3 = -6/8 = -0.75$. The other derivative is $\partial Q/\partial p_2 = 18p_1^{-1/2}p_2^{1/2}$ which takes the value $(18/6)3 = 9$. The change in Q is approximately $-0.75(4)+9(-.5) = -7.5$.

c) If both prices increase by 0.4, the estimated change in demand is $-0.75(0.4) + 9(0.4) = +3.3$.

d) The actual numbers are approximately $Q = 12(40)^{-1/2}(8.5)^{3/2} \approx 47.02$, which is a fall of 6.92, compared with the estimated fall of 7.5 in case (b). In case (c), $Q = 12(36.4)^{-1/2}(9.4)^{3/2} \approx 57.32$ compared with an estimated 57.30.

3. Let $f(x) = x^4 - 4x^3 + 4x^2 + 1$. Find all of the critical points and classify them (e.g. strict local max). Are there any global maxima or minima?

Answer: We solve $f'(x) = 4x^3 - 12x^2 + 8x = 0$ to find the critical points. They are $x = 0$, $x = 1$, and $x = 2$. Now $f''(x) = 12x^2 - 24x + 8$, so the second derivatives are $f''(0) = 8$, $f''(1) = -4$, and $f''(2) = 8$. Based on the second deriviate test, critical points at $x = 0$ and $x = 2$ are strict local minima and the critical point at $x = 1$ is a strict local maximum.

As $x \to \pm\infty$, $f(x) \to +\infty$, so there is no global maximum. Now $f(0) = 1$ and $f(2) = 1$, so both $x = 0$ and $x = 2$ are strict local minima and non-strict global minima. Since there are two of them, they cannot be globally strict.

4. Consider the quadratic form $Q(x,y) = x^2 + 2xy + 3y^2 - 2xz + z^2$.

a) Is the form positive or negative definite? Semidefinite? Indefinite?

b) Suppose we constrain the form to the subspace $x+3y-z = 0$. Does it have a unique maximum or minimum on this subspace?

Answer:

a) The form can be generated from the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

The leading principal minors are $\det \mathbf{A}_1 = 1$, $\det \mathbf{A}_2 = 3 - 1 = 2$, and $\det \mathbf{A} = 1(3) - 1 - (3) = -1$. With the first two positive and the last negative, it violates both sign patterns. The quadratic form is **indefinite**.

b) Form the bordered Hessian

$$\mathsf{H} = \begin{pmatrix} 0 & 1 & 3 & -1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 3 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}.$$

Here there are $m = 3$ variables and $k = 1$ constraint, so we must check the last 2 leading principal minors. They are $\det \mathsf{H}_3 = -6$ and $\det \mathsf{H}_4 = 1$. These have opposite signs, so it is not constrained positive definite. Also, $(-1)^3 \det \mathsf{H} = -1 < 0$, so it is not constrained negative definite either. It is **constrained indefinite**. In fact, both $(0,1,3)$ and $(-10,1,-7)$ are in the constraint set with $Q(0,1,3) = 12$ and $Q(-10,1,-7) = -8$.