Mathematical Economics Final, December 7, 2021

1. Maximize the utility function $u(x, y) = -e^{-2x} - e^{-3y}$ subject to the budget constraints $4x + y \leq 10$, $x \geq 0$, $y \geq 0$. Don't forget to consider constraint qualification and the second order conditions.

Answer: Form the Lagrangian

$$\mathcal{L} = -e^{-2x} - e^{-3y} - \mu_0(4x + y - 10) + \mu_x x + \mu_y y.$$

Since this is a standard consumer's problem, constraint qualification will be satisfied. This happens because the derivative of the constraint vector is

$$\begin{pmatrix} 4 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

At most two constraints can bind, and the derivative of the binding constraints will have maximum rank if either one or two constraints binds.

We also note the second order sufficient conditions for a maximum are satisfied because the objective has Hessian

$$H = \begin{pmatrix} -4e^{-2x} & 0 \\ 0 & -9e^{-3y} \end{pmatrix}$$

which is negative definite as a negative diagonal matrix. When constraints bind, this automatically implies the Hessian restricted to the tangent space is negative definite, so we don't have to worry about using the bordered Hessian.

The maximum must obey the first order conditions

$$0 = 2e^{-2x} - 4\mu_0 + \mu_x$$
$$0 = 3e^{-3y} - \mu_0 + \mu_y$$

for some $\mu_0, \mu_x \mu_y \geq 0$. The first order equations both imply $\mu_0 > 0$, so $4x + y = 10$.

If $x = 0$, $y = 10$ and $\mu_y = 0$ by complementary slackness. The second FOC implies $\mu_0 = 3e^{-10}$. But then $\mu_x = -2 + 12e^{-10} \approx -2 < 0$ by the first FOC, which is impossible.

If $y = 0$, $x = 5/2$ and $\mu_x = 0$ by complementary slackness. Then $\mu_0 = (1/2)e^{-5}$, implying $\mu_y - 3 + (1/2)e^{-5} < 0$, which is impossible.

It must be that $x, y > 0$, when $\mu_x = \mu_y = 0$ by complementary slackness. In that case $(1/2)e^{-2x} = \mu_0 = 3e^{-3y}$. Taking the logarithm and rearranging yields $3y = 2x + \ln 6$. We now use the budget constraint to find $30 - 14x = \ln 6$, when

$$x = \frac{30 - \ln 6}{14} \approx 2.015 > 0$$
$$y = \frac{10 + 2\ln 6}{7} \approx 1.94 > 0,$$

which is the solution.

2. Find the point on the parabola defined by $y = x^2 + 1$ that is closest to $(3, 1)$.

Answer: The derivative of the constraint is $(2x, -1)$, which always has rank one, so NDCQ is satisfied.

We minimize the squared distance $(x - 3)^2 + (y - 1)^2$ subject to the constraint $y = x^2 + 1$. We use the squared distance to make the calculations easier.

The Lagrangian is

$$\mathcal{L} = (x - 3)^2 + (y - 1)^2 - \lambda(y - x^2 - 1).$$

The first order conditions are

$$0 = 2(x - 3) + 2\lambda x$$
$$0 = 2(y - 1) - \lambda$$

Now $y - 1 = x^2$, so the second equation becomes $2x^2 = \lambda$. Simplify the first FOC to $x - 3 + \lambda x = 0$ and substitute $\lambda = x^2$ to obtain $0 = 2x^3 + x - 3$. This obviously has root $x = 1$. We factor it to $0 = (x - 1)(2x^2 + 2x + 3)$ The other roots are complex, and we need not consider them. Since $x = 1$, $y = 2$ by the constraint. There is only one critical point, $(1, 2)$.

The closest point is $(1, 2)$.

3. Consider the following second order homogeneous differential equation with constant coefficients:

$$\ddot{y} - \dot{y} - 20y = 0$$

a) Find and solve the characteristic equation.

b) State the general solution.

c) Show that the equation has a solution for any initial data $y_0 = y(0)$ and $y_1 = \dot{y}(0)$.

Answer:

a) We substitute $y(t) = e^{rt}$ to find the characteristic equation. That yields $(r^2 - r - 20)e^{rt} = 0$, so the characteristic equation is

$$r^2 - r - 20 = 0.$$

It has two solutions, $r = 5$ and $r = -4$.

b) The general solution is then $c_1 e^{5t} + c_2 e^{-4t}$.

c) Using the general solution, we find $c_1 + c_2 = y_0$ and $5c_1 - 4c_2 = y_1$. This can be written in matrix form as

$$\begin{pmatrix} 1 & 2 \\ 5 & -4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}.$$

Since the determinant of the matrix is $-14 \neq 0$ the equation will have a unique solution (c_1, c_2) for any (y_0, y_1).

4. Consider the set defined by $M = \{(x, y, z) : x^2 + y^2 - z^2 = 1\}$. Is M a 2 dimensional manifold? I.e., if $(x_0, y_0, z_0) \in M$ is arbitrarily chosen, does the Implicit Function Theorem define one of the variables as a function of the other two on a neighborhood of (x_0, y_0, z_0)?

Answer: We compute the derivative of $f(x, y, z) = x^2 + y^2 - z^2$,

$$Df = (2x, 2y, -2z).$$

Now at least one of x and y must be non-zero because $x^2 + y^2 = 1 + z^2 \geq 1$. Either way, Df will have rank one, meaning that f is regular. By the Implicit Function Theorem, whichever of x and y has a non-zero component can now be written in terms of the other variable and z in a neighborhood of (x_0, y_0, z_0). Which variable is a function of the others may depend on our starting point (x_0, y_0, z_0).

See also Theorem 15.26.1 in the notes for additional details.

5. Maximize $u(x, y) = 2x + 3y$ subject to the constraints $2x + y \leq 10$, $x + 2y \leq 10$, $x \geq 0$, $y \geq 0$.

Answer: We start by writing the Lagrangian:

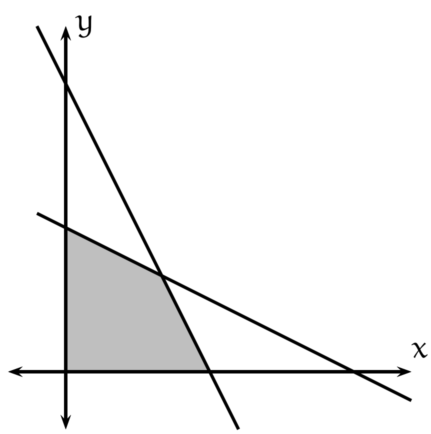
$$\mathcal{L} = 2x + 3y - \lambda_1(2x + y - 10) - \lambda_2(x + 2y - 10) + \mu_x x + \mu_y y.$$

This is a non-standard consumer's problem, so we take a quick look at constraint qualification. An easy way to get constraint qualification is to note that all the constraints are linear, which is another method of constraint qualification.

NDCQ also works. The derivative of the constraint vector is

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

The figure illustrates the four constraints, with the shaded area being our non-standard budget set. It's clear that at most two constraints can bind at once. Since any two lines of the matrix are linearly independent, NDCQ is satisfied. By the way, the linear independence is evident in the figure too—it follows from the fact that none of the four constraint lines are parallel.



The Hessian is the zero matrix, implying that the objective is both globally convex and concave. But we already knew that because it is linear. Of course, it is also convex and concave on any tangent space defined by binding constraints. Any critical point will be a maximum (but not necessarily strict).

The first order conditions are

$$\begin{aligned} 0 &= 2 - 2\lambda_1 - \lambda_2 + \mu_x \\ 0 &= 3 - \lambda_1 - 2\lambda_2 + \mu_y. \end{aligned}$$

With a linear objective and constraint, this will be an exercise in complementary slackness. We can rewrite the first order conditions as

$$\begin{aligned} 2\lambda_1 + \lambda_2 &= 2 + \mu_x \\ \lambda_1 + 2\lambda_2 &= 3 + \mu_y. \end{aligned} \tag{1}$$

Adding equations (1) together, we obtain $3(\lambda_1 + \lambda_2) = 5 + \mu_x + \mu_y \geq 5$. At least one of λ_1, λ_2 must be positive.

If both are positive, then both $x + 2y = 10$ and $2x + y = 10$ by complementary slackness. We solve the equations, finding $x = y = 10/3$. It follows that $\mu_x = \mu_y = 0$ by complementary slackness. Then we can solve for $\lambda_1 = 4/3$, $\lambda_2 = 2/3$, showing that we have a critical point at $(x, y) = (10/3, 10/3)$.

But what if $2x + y = 10$ and $x + 2y < 10$? Then $\lambda_2 = 0$ and we have

$$\begin{aligned} 2\lambda_1 &= 2 + \mu_x \\ \lambda_1 &= 3 + \mu_y. \end{aligned} \tag{2}$$

Together, equations (2) show $2 + \mu_x = 2\lambda_1 = 6 + 2\mu_y \geq 6$, so $\mu_x > 0$. By complementary slackness, $x = 0$. Since $2x + y = 10$, we have $y = 10$, but that violates $x + 2y < 10$. There is no solution here.

Now suppose $2x + y < 10$ but $x + 2y = 10$, we have $\lambda_1 = 0$ and

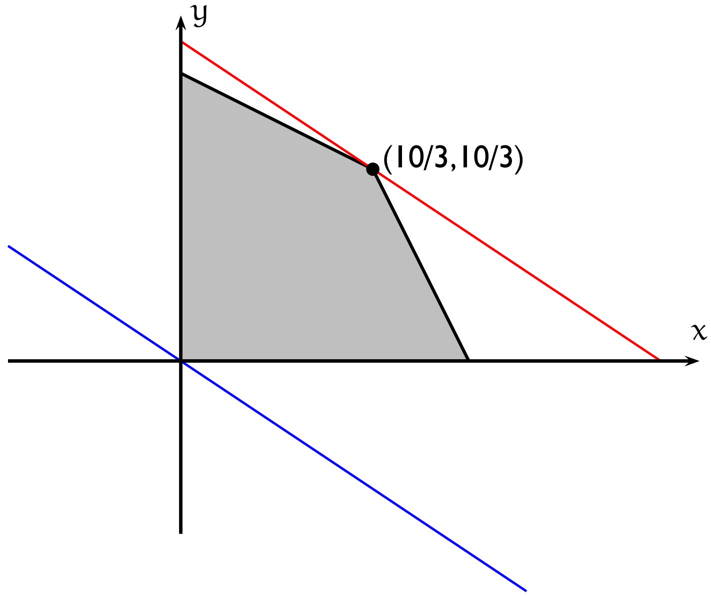
$$\begin{aligned} \lambda_2 &= 2 + \mu_x \\ 2\lambda_2 &= 3 + \mu_y. \end{aligned} \tag{3}$$

Together, equations (3) show $3 + \mu_y = 2\lambda_2 = 4 + 2\mu_x \geq 4$, so $\mu_y > 0$. By complementary slackness, $y = 0$. Since $x + 2y = 10$, we have $x = 10$. But that violates $2x + y < 10$. There is no solution here either.

It follows that $(x, y) = (10/3, 10/3)$ is the only critical point. As such, it must maximize utility. The point is that the feasible set is compact and the objective is continuous, so Weierstrass's Theorem guarantees there is a maximum.

The only critical point must be it and must be a strict maximum. If it were not, there would be more critical points.

The maximium is illustrated in the figure below where the red line is the indifference curve through the maximal point. I've also shown the solution to the minimization problem, where the blue line is the relevant indifference curve.



NB: If we were minimizing, the non-negativity conditions on the multipliers would be different and yield $(0,0)$ as the only critical point.

In fact, at $(x,y) = (0,0)$, the constraints $2x + y \leq 10$ and $x + 2y \leq 10$ do not bind, so $\lambda_1 = \lambda_2 = 0$ by complementary slackness. The first order conditions, equations (1), become $\mu_x = -2$ and $\mu_y = -3$. Combined with $\lambda_1 = \lambda_2 = 0$, the values of the multipliers indicate a minimum.

Note also that at the other two corners of the constraint set, the multipliers have mixed signs, indicating they are neither maxima nor minima.