Homework Assignment #1

6.3 The economy on the island of Bacchus produces only grapes and wine. The production of 1 pound of grapes requires 1/2 pound of grapes, 1 laborer, and 7/8 liter of wine. The production of 1 liter of wine requires 1/2 pound of grapes, 1 laborer, and 1/4 liter of wine. The island has 10 laborers who all together demand 1 pound of grapes and 3 liters of wine for their own consumption. Write out the input-output system for the economy of this island. Can you solve it?

Answer: Let g and w respectively denote the amount of grapes and wine produced. Demand for grapes from wine production is $w/2$, demand from grape production is $g/2$ and demand for consumption is 1. Since demand must equal supply, $g = g/2 + w/2 + 1$. For wine, the corresponding equation is $w = w/4 + 3$. Finally, for labor, $10 = w + g$. This yields the following input-output system:

$$\begin{aligned} g/2 - w/2 &= 1 \\ 3w/4 &= 3 \\ w + g &= 10 \end{aligned}$$

This is easily solved since the second equation implies $w = 4$. Then the third equation yields $g = 6$, and we have merely to verify that the first equation is also satisfied (otherwise there would be no solution).

6.5 Suppose that 10 percent of white males of working age and 20 percent of black males of working age are unemployed right now. According to Hall's model, what will the corresponding unemployment rates be in the next period?

Answer: Hall's model is given by equations (5) and (6) on page 114. For white males, $x_t = .9$ and $y_t = .1$. Using equations (5), we find that the unemployment rate drops to 8.82%. For black males, $x_t = .8$ and $y_t = .2$. Equations (6) predict the black male unemployment rate drops to 18.28%.

Write the three systems in Exercise 7.3 in matrix form. Then use row operations to find their corresponding row echelon and reduced row echelon forms and to find the solution.

Answer:

a)

$$\begin{pmatrix} 3 & 3 & | & 4 \\ 1 & -1 & | & 10 \end{pmatrix} \xrightarrow{(2)-3(1)} \begin{pmatrix} 3 & 3 & | & 4 \\ 0 & -2 & | & 26/3 \end{pmatrix}$$

yields row echelon form. Then dividing the second row by -2, we continue until the row echelon form is obtained.

$$\begin{pmatrix} 3 & 3 & | & 4 \\ 0 & 1 & | & -13/3 \end{pmatrix} \xrightarrow{(1)/3} \begin{pmatrix} 1 & 1 & | & 4/3 \\ 0 & 1 & | & -13/3 \end{pmatrix} \xrightarrow{(1)-(2)} \begin{pmatrix} 1 & 0 & | & 17/3 \\ 0 & 1 & | & -13/3 \end{pmatrix}$$

so the solution is $(17/3, -13/3)^T$.

b)

$$\begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 6 & 3 & -5 & | & 0 \\ 1 & 1 & 2 & | & 9 \end{pmatrix} \xrightarrow{(2)-3(1)/2} \begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 0 & 0 & -1/2 & | & -3/2 \\ 1 & 1 & 2 & | & 9 \end{pmatrix} \xrightarrow{(3)-(1)/4} \begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 0 & 0 & -1/2 & | & -3/2 \\ 0 & 1/2 & 11/4 & | & 35/4 \end{pmatrix}$$

We now interchange rows (2) and (3) to get row echelon form and continue reducing.

$$\begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 0 & 1/2 & 11/4 & | & 35/4 \\ 0 & 0 & 1/2 & | & -3/2 \end{pmatrix} \xrightarrow{2(2)} \begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 0 & 1 & 11/2 & | & 35/2 \\ 0 & 0 & -1/2 & | & -3/2 \end{pmatrix} \xrightarrow{-2(3)} \begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 0 & 1 & 11/2 & | & 35/2 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

$$\xrightarrow{1(4)} \begin{pmatrix} 1 & 1/2 & -3/4 & | & 1/4 \\ 0 & 1 & 11/2 & | & 35/2 \\ 0 & 0 & 1 & | & 3 \end{pmatrix} \xrightarrow{(1)-(2)/2} \begin{pmatrix} 1 & 0 & -7/2 & | & -17/2 \\ 0 & 1 & 11/2 & | & 35/2 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

$$\xrightarrow{(1)+7(3)/2} \begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 11/2 & | & 35/2 \\ 0 & 0 & 1 & | & 3 \end{pmatrix} \xrightarrow{(2)-11(3)/2} \begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

which is the reduced row echelon form. The solution is $(2, 1, 3)^T$.

c)

$$\begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 1 & 1 & 1 & | & -2 \\ 2 & -4 & 3 & | & 0 \end{pmatrix} \xrightarrow{(3)-(1)} \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 1 & 1 & 1 & | & -2 \\ 0 & -6 & 4 & | & -2 \end{pmatrix} \xrightarrow{(2)-(1)/1} \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 0 & 0 & 3/2 & | & -3 \\ 0 & -6 & 4 & | & -2 \end{pmatrix}$$

Again, we switch rows (2) and (3) to obtain the row echelon form, then continue.

$$\begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 0 & -6 & 4 & | & -2 \\ 0 & 0 & 3/2 & | & -3 \end{pmatrix} \xrightarrow{(1)/2} \begin{pmatrix} 1 & 1 & -1/2 & | & 1 \\ 0 & -6 & 4 & | & -2 \\ 0 & 0 & 3/2 & | & -3 \end{pmatrix} \xrightarrow{-(2)/6} \begin{pmatrix} 1 & 1 & -1/2 & | & 1 \\ 0 & 1 & -2/3 & | & 1/3 \\ 0 & 0 & 3/2 & | & -3 \end{pmatrix}$$

$$\xrightarrow{2(3)/3} \begin{pmatrix} 1 & 1 & -1/2 & | & 1 \\ 0 & 1 & -2/3 & | & 1/3 \\ 0 & 0 & 1 & | & -2 \end{pmatrix} \xrightarrow{(1)+(3)/2} \begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & -2/3 & | & 1/3 \\ 0 & 0 & 1 & | & -2 \end{pmatrix}$$

$$\xrightarrow{(2)+2(3)/3} \begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & -2 \end{pmatrix} \xrightarrow{(1)-(2)} \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & -2 \end{pmatrix}$$

which is the reduced row echelon form. The solution is $(1, -1, -2)^T$.

7.18 For what values of the parameter a does the following system of equations have a solution?

$$\begin{aligned} 6x + y &= 7 \\ 3x + y &= 4 \\ -6x - 2y &= a. \end{aligned}$$

Answer: We form the augmented matrix and do some row reduction.

$$\hat{A} = \begin{pmatrix} 6 & 1 & 7 \\ 3 & 1 & 4 \\ -6 & -2 & a \end{pmatrix} \xrightarrow{(2)-(1)/2,(3)+(1)} \begin{pmatrix} 6 & 1 & 7 \\ 0 & 1/2 & 1/2 \\ 0 & -1 & a+7 \end{pmatrix} \xrightarrow{2(2)} \begin{pmatrix} 6 & 1 & 7 \\ 0 & 1 & 1 \\ 0 & -1 & a+7 \end{pmatrix} \xrightarrow{(3)+(2)} \begin{pmatrix} 6 & 1 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & a+8 \end{pmatrix}$$

This system will only have a solution if the augmented matrix and the coefficient matrix have the same rank. It is clear that $\text{rank}(A) = 2$ and that $\text{rank}(\hat{A}) = 2$ if and only if $a + 8 = 0$, i.e., $a = -8$.

7.20 Compute the rank of each of the following matrices:

$$\text{a)} \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \quad \text{b)} \begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \quad \text{c)} \begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{pmatrix},$$

$$\text{d)} \begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 1 & 9 & -6 & 4 & 9 \\ 1 & 3 & -8 & 4 & 2 \\ 2 & 15 & -13 & 11 & 16 \end{pmatrix}, \quad \text{e)} \begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{pmatrix}.$$

Answer: We row-reduce the matrices to determine the rank. In case (a), the rank is 1, the rank of (b) is 2, the rank of (c) is 3, the rank of (d) is 3, and the rank of (e) is 3.