Homework Assignment #2

8.15 Check that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} .5 & 0 & -.5 \\ .5 & 0 & .5 \\ -.5 & 1 & -.5 \end{pmatrix}.$$

Answer: Computation shows that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and that

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} .5 & 0 & -.5 \\ .5 & 0 & .5 \\ -.5 & 1 & -.5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} .5 & 0 & -.5 \\ .5 & 0 & .5 \\ -.5 & 1 & -.5 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

8.28 What is the inverse of the $n \times n$ diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{pmatrix}?$$

Answer: Provided $d_i \neq 0$ for $i = 1, \dots, n$, the inverse is

$$D^{-1} = \begin{pmatrix} 1/d_1 & 0 & 0 & \dots & 0 \\ 0 & 1/d_2 & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1/d_n \end{pmatrix}.$$

If any $d_i = 0$, there is no inverse.

8.35 Let the general 2×2 technology matrix be given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Prove Theorem 8.13 directly for such a matrix using Theorem 8.8.

Answer: Consider $B = I - A$. Then

$$B = \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}.$$

By Theorem 8.8, this is invertible if and only if $(1-a)(1-d) - bc \neq 0$. Since columns sum to something less than 1, $1 - a > c$ and $1 - d > b$. Because b and c are positive, we multiply and conclude $(1-a)(1-d) > bc$. Thus $(1-a)(1-d) - bc > 0$, so B is invertible.

Also by Theorem 8.8,

$$B^{-1} = \frac{1}{(1-a)(1-d) - bc} \begin{pmatrix} 1-d & b \\ c & 1-a \end{pmatrix}.$$

The denominator is positive, and the entries are positive, so the matrix is positive.

9.9 Use Theorem 9.3 to determine which of the matrices in Exercises 9.7 and 9.8 are nonsingular.

Answer: We have to check whether the determinant is zero.

a) The first matrix in 9.7 is $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, which has determinant $1 - 2 = -1$. It is nonsingular.

b) The second matrix in 9.7 is $\begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}$, which has determinant $-72 + 60 = -12$. It is nonsingular.

c) The third matrix in 9.7 is $\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & 7 & 8 \end{pmatrix}$, which has determinant $42 - 24 = 18$. It is nonsingular.

d) The first matrix in 9.8 is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{pmatrix}$, which has determinant $12 + 2 + 4 - 4 - 8 - 3 = 3$. It is nonsingular.

e) The second matrix in 9.8 is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 1 & 9 & 6 \end{pmatrix}$, which has determinant $24 + 5 - 4 - 45 = -20$. It is nonsingular.

26.13

a) Compute the determinant of each of the following matrices by applying row operations to obtain an upper-triangular matrix and then use Fact 26.11:

$$\text{i)}\ \begin{pmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{pmatrix}, \quad \text{ii)}\ \begin{pmatrix} 2 & 3 & 1 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 2 & 4 \\ 4 & 6 & 2 & 0 \end{pmatrix}, \quad \text{iii)}\ \begin{pmatrix} 2 & 6 & 0 & 5 \\ 6 & 211 & 8 & 17 \\ 4 & 12 & -4 & 13 \\ 0 & -3 & 12 & 2 \end{pmatrix}.$$

b) Which of these matrices are nonsingular?

Answer:

a)

i) We start by row-reducing:

$$\begin{pmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 6 \\ 0 & -1 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix}.$$

The determinant is -6.

ii) Again, we row-reduce:

$$\begin{pmatrix} 2 & 3 & 1 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 2 & 4 \\ 4 & 6 & 2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 3 & 1 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

which has determinant 8.

iii) Once again, we row-reduce:

$$\begin{pmatrix} 2 & 6 & 0 & 5 \\ 6 & 21 & 8 & 17 \\ 4 & 12 & -4 & 13 \\ 0 & -3 & 12 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 6 & 0 & 5 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & -4 & 3 \\ 0 & -3 & 12 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 6 & 0 & 5 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & 20 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 6 & 0 & 5 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & 19 \end{pmatrix}.$$

which has determinant -456.

b) All have non-zero determinants, so all are nonsingular.