Homework Assignment #3

12.6 Prove that if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences with limits x and y, respectively, then the sequence $\{x_n - y_n\}_{n=1}^{\infty}$ converges to the limit $x - y$.

Answer: Consider

$$\begin{aligned} |(x_n - y_n) - (x - y)| &= |(x_n - x) - (y_n - y)| \\ &\leq |x_n - x| + |y_n - y|. \end{aligned}$$

Let $\varepsilon > 0$. Choose N_1 such that $|x_n - x| < \varepsilon/2$ for $n > N_1$. Then choose $N_2 \geq N_1$ with $|y_n - y| < \varepsilon/2$ for $n > N_2$. It follows that for $n > N_2 \geq N_1$,

$$|(x_n - y_n) - (x - y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

establishing convergence.

12.9 A sequence is said to be **bounded** if there is a number B such that $|x_n| \leq B$ for all n. Show that if $\{x_n\}_{n=1}^{\infty}$ converges to 0 and if $\{y_n\}_{n=1}^{\infty}$ is bounded, then the product sequence converges to 0.

Answer: Let B be a bound for $\{y_n\}$ and let $\epsilon > 0$. Take N such that $|x_n| < \epsilon/B$ for $n \geq N$. Then $|x_n y_n| \leq B|x_n| < B\epsilon/B = \epsilon$ for $n \geq N$. This establishes that the product converges to zero.

12.18 Show that *closed intervals* in $\mathbb{R}^1$ — sets of the form $\{x : a \leq x \leq b\}$ for fixed numbers a and b — are closed sets.

Answer: Let $[a, b]$ be a closed interval and $x_n \in [a, b]$ with $x_n \to x$. We must show $x \in [a, b]$. By Theorem 12.4, $a \leq x \leq b$, so $x \in [a, b]$. Therefore $[a, b]$ is closed.

12.23 Suppose that S is a subset of $\mathbb{R}^n$ with complement T. Show that cl S is the complement of int T.

Answer: Suppose $x \in \text{cl}\, S$. Then either $x \in S$ or there is a sequence $\{x_n\}$ of points in S with $x_n \to x$. In the first case, $x \notin T$, so $x \notin \text{int}\, T \subset T$. In the second case, we can also show $x \notin T$. If it $x \in T$, then there would be $\varepsilon > 0$, with $B_\varepsilon(x) \subset \text{int}\, T$. But then $x_n \in B_\varepsilon(x)$ for large n, meaning that $x_n \in \text{int}\, T \subset T$ for large n. But then $x_n \notin S$ for large n. This contradiction shows $x \notin T$. It follows that if $x \in \text{cl}\, S$, $x \notin \text{int}\, T$. In other words, $\text{cl}\, S \subset (\text{int}\, T)^c$.

Now suppose $x \notin \text{int}\, T$. In this case, for every $n > 0$, there is $x_n \in B_{1/n}(vx)$ with $x_n \notin T$. It follows that $x_n \in S$. Now $x_n \to x$, showing that x is a limit point of S, and so in cl S. It follows that $(\text{int}\, T)^c \subset \text{cl}\, S$.

Combining the two inclusions shows $\text{cl}\, S = (\text{int}\, T)^c$.

27.1 Which of the following are subspaces of $\mathbb{R}^2$? Explain your answer.

a) $\{(x,y) : x = 0\}$, b) $\{(x,y) : x = 1\}$, c) $\{(x,y) : 3x - y = 0\}$,

d) $\{(x,y) : x^2 = y^2\}$, e) $\{(0,1)\}$, f) $\{(x,y) : x + y = 0, x - y = 0\}$

Answer: The sets (a), (c), and (f) are subspaces, the others are not subspaces. For (a) if (x,y) and (x',y') are in the set, $x = x' = 0$. Then $\alpha(x,y) + \beta(x',y') = (0, \alpha y + \beta y')$, which is in the set.

For (b), $(1,0)$ and $(1,2)$ are in the set, but $(1,0) + (1,2) = (2,2)$ is not in the set.

For (c), for (x,y) and (x',y') in the set, $3x = y$ and $3x' = y'$. Now $\alpha(x,3x) + \beta(x',3x') = \big((\alpha x + \beta x'), 3(\alpha x + \beta x')\big)$ is also in the set.

For (d), $(1,1)$ and $(1,-1)$ are in the set but $(1,1) + (1,-1) = (2,0)$ is not in the set.

For (e), $2(0,1) = (0,2)$ is not in the set.

For (f), if (x,y) is in the set, $x + y = 0$ and $x - y = 0$. Together these imply $(x,y) = (0,0)$. Then $\alpha(0,0) + \beta(0,0) = (0,0)$, so it is a linear subspace.