Mathematical Economics Exam #1, September 23, 2021

1. Consider the linear system

$$\begin{aligned} x + 2y + 3z &= a \\ 2x + 2y + 3z &= b \\ 3x + 4y + 6z &= c \end{aligned}$$

   a) For what values of a, b, c does this system have a solution?

   b) When the system has a solution, when is that solution unique?

   Answer:

   a) We start by forming the augmented matrix and row-reducing.

$$\begin{pmatrix} 1 & 2 & 3 & a \\ 2 & 2 & 3 & b \\ 3 & 4 & 6 & c \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & a \\ 2 & 2 & 3 & b \\ 0 & 0 & 0 & c-a-b \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & a \\ 1 & 0 & 0 & b-a \\ 0 & 0 & 0 & c-a-b \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 0 & b-a \\ 1 & 2 & 3 & a \\ 0 & 0 & 0 & c-a-b \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & b-a \\ 0 & 2 & 3 & 2a-b \\ 0 & 0 & 0 & c-a-b \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & b-a \\ 0 & 1 & 3/2 & a-b/2 \\ 0 & 0 & 0 & c-a-b \end{pmatrix}$$

   This will have a solution if and only if $c = a + b$.

   b) When $c = a + b$, the system has solutions. They can be written

$$\begin{aligned} x &= b - a \\ y &= a - \frac{b}{2} - \frac{3}{2}z. \end{aligned}$$

   Here z is a free variable, so there are infinitely many solutions. Another way to see this is that $\operatorname{rank} \mathbf{A} = 2$ which is less than the number of variables ($n = 3$).

2. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 8 & 27 & 4 \\ 2 & 4 & 9 & 16 \end{pmatrix}$$

   a) What is the rank of $\mathbf{A}$?

   b) Recall $\ker \mathbf{A} = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{0}\}$. What is $\dim \ker \mathbf{A}$?

   c) Find a basis for $\ker \mathbf{A}$.

Answer:

a) We start by row-reducing $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 8 & 27 & 4 \\ 2 & 4 & 9 & 16 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 24 & 0 \\ 0 & 0 & 3 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 8/3 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 0 & -5 & 4 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 8/3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 52/3 \\ 0 & 1 & 0 & -32/3 \\ 0 & 0 & 1 & 8/3 \end{pmatrix}$$

which has rank 3.

b) By the Fundamental Theorem of Linear Algebra, the $\operatorname{rank}\mathbf{A} + \dim\ker\mathbf{A} = 4$, so $\dim\ker\mathbf{A} = 1$.

Alternatively, there is one free variable, so $\dim\ker\mathbf{A} = 1$.

c) The kernel solves $\mathbf{A}\mathbf{x} = \mathbf{0}$, or

$$\begin{aligned} x_1 &= -(52/3)x_4 \\ x_2 &= (32/3)x_4 \\ x_3 &= -(8/3)x_4 \end{aligned}$$

Any vector in $\ker\mathbf{A}$ will be a non-zero multiple of

$$\mathbf{b} = \begin{pmatrix} -52/3 \\ 32/3 \\ -8/3 \\ 1 \end{pmatrix},$$

which we obtained by setting $z_4 = 1$. Then $\{\mathbf{b}\}$ is a basis for $\ker\mathbf{A}$.

3. Let V be an inner product space and $\mathbf{x}, \mathbf{y} \in V$ with $\mathbf{x} \neq \mathbf{y}$. Show that

$$\mathbf{z} = \mathbf{y} - \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|^2}\mathbf{x}$$

is perpendicular to $\mathbf{x}$.

Answer: I will assume we are dealing with a real vector space (as you did), so that $\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$. It works for complex spaces too. We compute

$$\begin{aligned} \mathbf{z}\cdot\mathbf{x} &= \mathbf{y}\cdot\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|^2}\mathbf{x}\cdot\mathbf{x} \\ &= \mathbf{y}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{y} \\ &= 0, \end{aligned}$$

Then $\mathbf{z}$ and $\mathbf{x}$ are perpendicular since their inner product is zero.

In the complex case the calculation is slightly different because $\mathbf{x}\cdot\mathbf{y} = \overline{\mathbf{y}\cdot\mathbf{x}}$ and the linear product is sesqui-linear. Here we would write

$$\begin{aligned} \mathbf{z}\cdot\mathbf{x} &= \mathbf{y}\cdot\mathbf{x} - \frac{\overline{\mathbf{x}\cdot\mathbf{y}}}{\|\mathbf{x}\|^2}\mathbf{x}\cdot\mathbf{x} \\ &= \mathbf{y}\cdot\mathbf{x} - \overline{\mathbf{x}\cdot\mathbf{x}} \\ &= \mathbf{y}\cdot\mathbf{x} - \mathbf{y}\cdot\mathbf{x} \\ &= 0, \end{aligned}$$

4. Consider the following norms on $\mathbb{R}^3$. The Euclidean norm $\|\mathbf{x}\|_2 = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, and the sup-norm $\|\mathbf{x}\|_\infty = \max\left\{|x_1|, |x_2|, |x_3|\right\}$.

   a) Find a number A so that $\|\mathbf{x}\|_2 \le A\|\mathbf{x}\|_\infty$.

   b) What do you expect A to be in $\mathbb{R}^n$?

Answer:

a) For each i, $|x_i| \le \|\mathbf{x}\|_\infty$, so $x_i^2 \le \|\mathbf{x}\|_\infty^2$. Then

$$\begin{aligned} \|\mathbf{x}\|_2 &= (x_1^2 + x_2^2 + x_3^2)^{1/2} \\ &\le \left(\|\mathbf{x}\|_\infty^2 + \|\mathbf{x}\|_\infty^2 + \|\mathbf{x}\|_\infty^2\right)^{1/2} \\ &= 3^{1/2}\|\mathbf{x}\|_\infty. \end{aligned}$$

We may take $A = \sqrt{3}$.

b) In $\mathbb{R}^n$, $A = \sqrt{n}$.