Mathematical Economics Exam #2, October 28, 2021

1. Consider the equation $x^4 - x^2 + y^2 = 0$.

   a) For what values of (x, y) with $-1 \le x \le 1$ does this **not** define a regular curve?

   b) If the curve is regular, for which (x, y) with $x \in [-1, +1]$ can we write y as a function of x?

   c) Is $\{(x, y) : x^4 - x^2 + y^2 = 0\}$ a manifold?

   Answer: The curve looks rather like an infinity symbol and is called the lemniscate of Gerono (or of Huygens).

   a) We compute the derivative of $F(x, y) = x^4 - x^2 + y^2$. It is $DF = (4x^3 - 2x, 2y)$. If the derivative is zero, then $y = 0$ and $4x^3 - 2x = 0$. One solution is $(x, y) = (0, 0)$. If $x \neq 0$, we obtain $4x^2 - 1 = 0$. But this contradicts $y = 0$, which requires $x^4 - x^2 = 0$, and since $x \neq 0$, it requires $x^2 = 1$, contradicting $4x^2 = 1$. So the only non-regular point on the curve is $(0, 0)$.

   b) The Implicit Function Theorem allows us to write y as a function of x whenever $\partial F/\partial y = 2y \neq 0$. That is, when $y \neq 0$. Since $x^4 - x^2 + Y^2 = 0$, this happens only when $x = 0, \pm 1$.

   We can write x as a function of y whenever $\partial F/\partial x = 4x^3 - 2x \neq 0$. We cannot write x as a function of y at $(x, y) = (0, 0)$, nor at $x = \pm 1/\sqrt{2}$, which corresponds to $y = \pm 1/2$. This gives us four points: $(\pm 1/\sqrt{2}, 1/2)$ and $(\pm 1/\sqrt{2}, -1/2)$.

   c) Short answer: NO.

   Long answer. Any neighborhood U of $(0, 0)$ has four arms. Suppose such a U is homeomorphic to an interval (a, b) via φ. Let $c = \varphi(0, 0)$. If we remove $(0, 0)$, φ is a homeomorphism between $U \setminus \{(0, 0)\}$ and $(a, c) \cup (c, b)$. Now $U \setminus \{(0, 0)\}$ consists of 4 components. But $\varphi^{-1}(a, c)$ and $\varphi^{-1}(c, b)$ are both connected and can only cover two components. As $\varphi^{-1}\big((a, c) \cup (c, b)\big) = U \setminus \{(0, 0)\}$, this is impossible. It follows that the sets are not homeomorphic, and the curve is not a manifold at $(0, 0)$.

2. For each of the following spaces X and sets $S \subset X$, determine whether each subset S is an open, closed, connected, and/or compact subset of X.

   a) $X = \mathbb{R}^2$ with the usual topology. $S = \{(n, m) : \text{both } n \text{ and } m \text{ are integers}\}$.

   b) $X = \mathbb{R}^2$ with the usual topology. $S = \{(x, y) : x^2 + y^2 \ge 1, x^2 + y^2 < 10\}$.

   c) $X = (0, 1) \cup (2, 3)$ with the usual topology. The set S is the interval $(0, 1)$

Answer:

a) The set of lattice points S is not open because $B_{1/2}(0,0)$ contains points that are not lattice points. The set S is closed, because if $x_n \in S$ and $x_n \to x$, then there is an N with $\|x_n - x\| < 1/2$ for $n > N$. Since any two lattice points are separated by at least distance 1, there can be only one such lattice point. This shows $x_n = x_N$ for $n > N$. It follows that the limit $x = x_N$ is also a lattice point, and thus in S. THe set S is not connected since the open sets $U = \{x \in S : x_1 < 1\}$ and $V = \{x \in S : x_1 > 0\}$ disconnect S. Finally, S is not compact because it isn't bounded.

b) The annulus S is neither closed nor open because in includes the inner boundary (not open) but not the outer boundary (not closed). Since it is not closed, it is not compact. The set S is connected because there's a path in S between any two points. If we use polar coordinates, writing the points as $(x_i, y_i) = (r_i \sin\theta_i, r_i \cos\theta_i)$, we have $1 \le r_1, r_2 < 10$, and the path between them can be written

$$f(t) = \big((1-t)r_1 + tr_2, (1-t)\theta_1 + t\theta_2\big)$$

for $t \in [0,1]$. Note that $1 \le (1-t)r_1 + tr_2 < 10$, so $f(t) \in S$ for all $t \in [0,1]$.

c) Interestingly, the set (0, 1) is both open and closed in X. If $x_n \in S$ and $x_n \to x$ with $x \in X$, it can't be that $x \in (2,3)$, so $x \in (0,1)$, showing (0, 1) is closed. As an interval, X is connected. As for compactness, the sequence $x_n = 1/n in S$ has no convergent subsequence since $0 \notin X$. Thus S is not compact.

3. Consider the quadratic form on $\mathbb{R}^3$ defined by

$$Q(x) = y^2 - 2z^2 + 2xy - 2xz + 2yz.$$

a) Is Q positive definite, negative definite, or indefinite?

b) Suppose we impose add the constraint $x + 2z = 0$. Is there a constrained maximum or minimum at the origin?

Answer:

a) It is indefinite. We compute $Q(0,1,0) = 1$ and $Q(0,0,1) = -1$, showing that the quadratic form is indefinite.

b) The easy way to solve this is to substitute $x = -2z$ in Q, obtaining $y^2 + 2z^2 - 2xy = (y-z)^2 + z^2$. This is non-negative, and is only zero at $(y,z) = (0,0)$, implying $x = 0$.

Thus it is positive definite under the constraint and has a unique constrained minimum at $(0, 0, 0)$.

Alternatively, the form has the associated matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \end{pmatrix}$$

The bordered Hessian is

$$H = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & 1 & -2 \end{pmatrix}.$$

There are $m = 3$ variables and $k = 1$ constraints, so we must check the last $m-k = 2$ leading principal minors. Now $\det H_3 = -1$ and $\det H_4 = -1$. The minors have the same sign, so we do not have a maximum at zero. To be a minimum at zero, the signs must be the same, and the common sign must be the same as $(-1)^m = -1$. It is, so we have a constrained minimum at $(0, 0, 0)$.

4. Find the first 4 terms of the Taylor expansion of $\sin x$ about $x = 0$.

Answer: We first compute $\sin x$ and its first three derivatives at $x = 0$. $\sin 0 = 0$, $\cos 0 = 1$, $-\sin 0 = 0$, and $-\cos 0 = -1$. Two of the terms are zero, leaving us with

$$x - \frac{x^3}{3}$$

as the first four terms.

FYI. The first four non-zero terms are

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

and the power series, which converges for all x, is

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$