Mathematical Economics Final, December 6, 2022

1. Let $f(x,y) = (x^2 - y^2)^2$.

   a) Find all critical points on $\mathbb{R}^2$.

   Answer: The derivative is
   $$Df = 2(x^2 - y^2)(2x, -2y).$$
   The covector $(2x, -2y)$ is zero only at $(x,y) = (0,0)$. The derivative is also zero when $x^2 - y^2 = 0$. That is, if $x = \pm y$.

   b) Which critical points are maxima? Minima?

   Answer: The function $f(x,y) = (x^2 - y^2)^2$ is always non-negative. It takes the minimum value zero whenever $x = \pm y$, so every critical point is a minimum. There are no maxima. In fact, if $y = \alpha x$ for $\alpha \neq \pm 1$, $f(x, \alpha x) = (1 - \alpha^2)^2 x^4$, which increases monotonically toward $+\infty$ as $x \to +\infty$ (or as $x \to -\infty$).

   FYI: The Hessian is
   $$H = \begin{pmatrix} 12x^2 - 4y^2 & -8xy \\ -8xy & 12y^2 - 4x^2 \end{pmatrix}$$
   At the critical points, $x^2 = y^2$, so we can write the Hessian there as
   $$H(x, \pm y) = \begin{pmatrix} 8x^2 & \mp 8x^2 \\ \mp 8x^2 & 8x^2 \end{pmatrix}$$
   which is positive semidefinite for $x \neq 0$.

   The situation around $(0,0)$ is a bit complex with minima on the x-shaped set where $x^2 = y^2$ and indefinite Hessians everywhere else, as befits a function with no maximum and minima on the X.

2. Maximize $f(x,y,z) = x + y + \sqrt{z}$ subject to the constraints $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x + 2y + z \leq 5$.

   Answer: This is a consumer's problem with $Df = (1, 2, \frac{1}{2}z^{-1/2}) > (0,0,0)$. We know from the homework that the budget constraint binds. As a result, at most 3 constraints can bind.

   The derivative of the constraints is
   $$Dg = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Whether one, two, or three constraints bind, the matrix formed by the binding constraints has full rank, so NDCQ is satisfied.

The Lagrangian is

$$\mathcal{L} = x + y + \sqrt{z} - \lambda(x + 2y + z - 5) + \mu_x x + \mu_y y + \mu_z z.$$

The first order conditions are

$$\begin{aligned} 0 &= 1 - \lambda + \mu_x \\ 0 &= 1 - 2\lambda + \mu_y \\ 0 &= \frac{1}{2\sqrt{z}} - \lambda + \mu_z \end{aligned}$$

We rewrite these as

$$\begin{aligned} \lambda &= 1 + \mu_x \\ 2\lambda &= 1 + \mu_y \\ \lambda &= \frac{1}{2\sqrt{z}} + \mu_z \end{aligned}$$

The first line tells us that $\lambda \geq 1$. Then by the second line, $\mu_y \geq 1$, implying $y = 0$ by complementary slackness.

The remaining first order conditions are

$$\begin{aligned} \lambda &= 1 + \mu_x \\ \lambda &= \frac{1}{2\sqrt{z}} + \mu_z \end{aligned}$$

These can't be satisfied with $z = 0$, so $z > 0$ and $\mu_z = 0$. Then $z = 1/4\lambda^2$. If $\lambda = 1$, $z = 1/4$ and $x = 19/4$ by the "budget constraint". Then $f(x, y, z) = 19/4 + 1/\sqrt{4} = 21/4$.

If $\lambda > 1$, $\mu_x = \lambda - 1 > 0$, so $x = 0$ by complementary slackness. It follows that $z = 5$ and $f(x, y, z) = 1/\sqrt{5} \approx 0.45$, which is not the maximum.

The maximum is $21/4 = 5\frac{1}{4}$ at $(x, y, z) = (19/4, 0, 1/4)$.

3. Consider the function $f(x, y, z) = e^{x^2} + \sin xyz - z^3$ on the closed ball $B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$. Without doing any computations, explain why f has both a maximum and minimum on the ball B

Answer: The ball B is both closed and bounded, hence compact. The function is the sum of three continuous functions, and so is continuous. By the Weierstrass Theorem, f must have both a maximum and minimum on B.

4. Consider the quadratic form $Q(x, y, z) = x^2 + 2xy + y^2 + 2xz + z^2$.

   a) Find a matrix $\mathbf{A}$ that defines the quadratic form Q.

   Answer: The matrix
$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$
   defines Q.

   b) Does the quadratic form Q have a maximum, minimum, or is it indefinite at $(0, 0, 0)$ under the constraint $x + 2y + z = 0$?

   Answer: We form the bordered matrix
$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

   There are $m = 3$ variables and $k = 1$ constraints, as $m - k = 2$ we must check the signs of the last two (third and fourth) leading principal minors. As both $(-1)^m = -1$ and $(-1)^k = -1$, the last principal minor must be negative for the form to be constrained definite.

   We row reduce to obtain the minors.

$$\begin{vmatrix} 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \longrightarrow - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \longrightarrow - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -1 & 0 \end{vmatrix} \longrightarrow$$
$$- \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix} \longrightarrow - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

   Since we did not use the last row to operate on the other rows, we can now read off the last two leading principal minors: $\det \mathbf{B}_3 = -1$ and $\det \mathbf{B}_4 = +1$. As the last leading principal minor is positive, the quadratic form is constrained indefinite. The point (0, 0, 0) is a saddlepoint.

5. Maximize $f(x, y, z) = x + yz$ under the constraints $x^2 + y^2 + z^2 \leq 25$, $x, y, z \geq 0$.

Answer: Here

$$Dg = \begin{pmatrix} 2x & 2y & 2z \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

It's impossible for all four constraints to bind at once, and the various submatrices have full rank unless $xyz = 0$. As (0, 0, 0), $(0, y, 0)$ and $(0, 0, z)$ minimize the function f and it is possible to do better, we don't have to worry about them. The point $(x, 0, 0)$ with $x > 0$ yields rank 3, so it is not a problem either. That takes care of the potential problem cases.

The Lagrangian is

$$\mathcal{L} = x + yz - \lambda(x^2 + y^2 + z^2 - 25) + \mu_x x + \mu_y y + \mu_z z.$$

so the first order conditions are

$$\begin{aligned} 0 &= 1 - 2\lambda x + \mu_x \\ 0 &= z - 2\lambda y + \mu_y \\ 0 &= y - 2\lambda z + \mu_z \end{aligned}$$

or

$$\begin{aligned} 2\lambda x &= 1 + \mu_x \\ 2\lambda y &= z + \mu_y \\ 2\lambda z &= y + \mu_z \end{aligned}$$

By the first line, $2\lambda x \geq 1$, so both $\lambda > 0$ and $x > 0$. By complementary slackness, $\mu_x = 0$ and so $\lambda x = 1$. Also, the constraint $x^2 + y^2 + z^2 = 25$ must bind.

Multiply the second equation by y and the third by z. Then

$$\begin{aligned} 2\lambda y^2 &= yz + \mu_y y = yz \\ 2\lambda z^2 &= yz + \mu_z z = yz \end{aligned}$$

by complementary slackness. It follows that $y^2 = z^2$. As both y and z are non-negative, $y = z$, implying $2\lambda y^2 = yz = y^2$. If $y > 0$, we find $\lambda = 1/2$. Then $x = 1$ and $y = z = \sqrt{12}$. In that case $f(1, \sqrt{12}, \sqrt{12}) = 13$. **This is the maximum.**

If $y = z = 0$, $x = 5$ by complementary slackness and $f(5, 0, 0) = 5$. This is neither a maximum nor minimum because within any neighborhood of $(5, 0, 0)$ there are points obeying the constraints where f is larger, and others where f is smaller. Decreasing x to increase both y and z increases f, while decreasing x while leaving y and z at zero, decreases f.