1. Let \( f(x, y) = (x^2 - y^2)^2 \).
   
   a) Find all critical points on \( \mathbb{R}^2 \).
   
   **Answer:** The derivative is
   \[
   Df = 2(x^2 - y^2)(2x, -2y).
   \]
   
   The covector \((2x, -2y)\) is zero only at \((x, y) = (0, 0)\). The derivative is also zero when \(x^2 - y^2 = 0\). That is, if \(x = \pm y\).

   b) Which critical points are maxima? Minima?
   
   **Answer:** The function \( f(x, y) = (x^2 - y^2)^2 \) is always non-negative. It takes the minimum value zero whenever \( x = \pm y \), so every critical point is a minimum. There are no maxima. In fact, if \( y = \alpha x \) for \( \alpha \neq \pm 1 \), \( f(x, \alpha x) = (1 - \alpha^2)x^4 \), which increases monotonically toward \(+\infty\) as \( x \to +\infty \) (or as \( x \to -\infty \)).

   **FYI:** The Hessian is
   \[
   H = \begin{pmatrix}
   12x^2 - 4y^2 & -8xy \\
   -8xy & 12y^2 - 4x^2
   \end{pmatrix}
   \]
   
   At the critical points, \( x^2 = y^2 \), so we can write the Hessian there as
   \[
   H(x, \pm y) = \begin{pmatrix}
   8x^2 & \mp 8x^2 \\
   \mp 8x^2 & 8x^2
   \end{pmatrix}
   \]
   
   which is positive semidefinite for \( x \neq 0 \).

   The situation around \((0, 0)\) is a bit complex with minima on the \( x \)-shaped set where \( x^2 = y^2 \) and indefinite Hessians everywhere else, as befits a function with no maximum and minima on the \( x \).

2. Maximize \( f(x, y, z) = x + y + \sqrt{z} \) subject to the constraints \( x \geq 0 \), \( y \geq 0 \), \( z \geq 0 \), and \( x + 2y + z \leq 5 \).

   **Answer:** This is a consumer’s problem with \( Df = (1, 2, \frac{1}{2}z^{-1/2}) \) \( > (0, 0, 0) \). We know from the homework that the budget constraint binds. As a result, at most 3 constraints can bind.

   The derivative of the constraints is
   \[
   Dg = \begin{pmatrix}
   1 & 2 & 1 \\
   -1 & 0 & 0 \\
   0 & -1 & 0 \\
   0 & 0 & -1
   \end{pmatrix}
   \]
Whether one, two, or three constraints bind, the matrix formed by the binding constraints has full rank, so NDCQ is satisfied.

The Lagrangian is

\[ L = x + y + \sqrt{z} - \lambda(x + 2y + z - 5) + \mu_x x + \mu_y y + \mu_z z. \]

The first order conditions are

\[ 0 = 1 - \lambda + \mu_x \]
\[ 0 = 1 - 2\lambda + \mu_y \]
\[ 0 = \frac{1}{2\sqrt{z}} - \lambda + \mu_z \]

We rewrite these as

\[ \lambda = 1 + \mu_x \]
\[ 2\lambda = 1 + \mu_y \]
\[ \lambda = \frac{1}{2\sqrt{z}} + \mu_z \]

The first line tells us that \( \lambda \geq 1 \). Then by the second line, \( \mu_y \geq 1 \), implying \( y = 0 \) by complementary slackness.

The remaining first order conditions are

\[ \lambda = 1 + \mu_x \]
\[ \lambda = \frac{1}{2\sqrt{z}} + \mu_z \]

These can't be satisfied with \( z = 0 \), so \( z > 0 \) and \( \mu_z = 0 \). Then \( z = 1/4\lambda^2 \). If \( \lambda = 1 \), \( z = 1/4 \) and \( x = 19/4 \) by the “budget constraint”. Then \( f(x, y, z) = 19/4 + 1/\sqrt{4} = 21/4 \).

If \( \lambda > 1 \), \( \mu_x = \lambda - 1 > 0 \), so \( x = 0 \) by complementary slackness. It follows that \( z = 5 \) and \( f(x, y, z) = 1/\sqrt{5} \approx 0.45 \), which is not the maximum.

The maximum is \( 21/4 = 5\frac{1}{4} \) at \((x, y, z) = (19/4, 0, 1/4)\).

3. Consider the function \( f(x, y, z) = e^{x^2} + \sin{xy}z - z^3 \) on the closed ball \( B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\} \). Without doing any computations, explain why \( f \) has both a maximum and minimum on the ball \( B \).
Answer: The ball $B$ is both closed and bounded, hence compact. The function is the sum of three continuous functions, and so is continuous. By the Weierstrass Theorem, $f$ must have both a maximum and minimum on $B$.

4. Consider the quadratic form $Q(x, y, z) = x^2 + 2xy + y^2 + 2xz + z^2$.
   
   a) Find a matrix $A$ that defines the quadratic form $Q$.
   
   Answer: The matrix
   
   $$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$ 
   
   defines $Q$.

   b) Does the quadratic form $Q$ have a maximum, minimum, or is it indefinite at $(0, 0, 0)$ under the constraint $x + 2y + z = 0$?
   
   Answer: We form the bordered matrix
   
   $$B = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$ 
   
   There are $m = 3$ variables and $k = 1$ constraints, as $m - k = 2$ we must check the signs of the last two (third and fourth) leading principal minors. As both $(-1)^m = -1$ and $(-1)^k = -1$, the last principal minor must be negative for the form to be constrained definite.

   We row reduce to obtain the minors.

   $$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

   Since we did not use the last row to operate on the other rows, we can now read off the last two leading principal minors: $\det B_3 = -1$ and $\det B_4 = +1$. As the last
leading principal minor is positive, the quadratic form is constrained indefinite. The point \((0, 0, 0)\) is a saddlepoint.

5. Maximize \(f(x, y, z) = x + yz\) under the constraints \(x^2 + y^2 + z^2 \leq 25\), \(x, y, z \geq 0\).

**Answer:** Here

\[
D_g = \begin{pmatrix}
2x & 2y & 2z \\
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{pmatrix}
\]

It’s impossible for all four constraints to bind at once, and the various submatrices have full rank unless \(xyz = 0\). As \((0, 0, 0)\), \((0, y, 0)\) and \((0, 0, z)\) minimize the function \(f\) and it is possible to do better, we don’t have to worry about them. The point \((x, 0, 0)\) with \(x > 0\) yields rank 3, so it is not a problem either. That takes care of the potential problem cases.

The Lagrangian is

\[
\mathcal{L} = x + yz - \lambda(x^2 + y^2 + z^2 - 25) + \mu_x x + \mu_y y + \mu_z z.
\]

so the first order conditions are

\[
0 = 1 - 2\lambda x + \mu_x \\
0 = z - 2\lambda y + \mu_y \\
0 = y - 2\lambda z + \mu_z
\]

or

\[
2\lambda x = 1 + \mu_x \\
2\lambda y = z + \mu_y \\
2\lambda z = y + \mu_z
\]

By the first line, \(2\lambda x \geq 1\), so both \(\lambda > 0\) and \(x > 0\). By complementary slackness, \(\mu_x = 0\) and so \(\lambda x = 1\). Also, the constraint \(x^2 + y^2 + z^2 = 25\) must bind.

Multiply the second equation by \(y\) and the third by \(z\). Then

\[
2\lambda y^2 = yz + \mu_y y = yz \\
2\lambda z^2 = yz + \mu_z z = yz
\]
by complementary slackness. It follows that $y^2 = z^2$. As both $y$ and $z$ are non-negative, $y = z$, implying $2\lambda y^2 = yz = y^2$. If $y > 0$, we find $\lambda = 1/2$. Then $x = 1$ and $y = z = \sqrt{12}$. In that case $f(1, \sqrt{12}, \sqrt{12}) = 13$. This is the maximum.

If $y = z = 0$, $x = 5$ by complementary slackness and $f(5, 0, 0) = 5$. This is neither a maximum nor minimum because within any neighborhood of $(5, 0, 0)$ there are points obeying the constraints where $f$ is larger, and others where $f$ is smaller. Decreasing $x$ to increase both $y$ and $z$ increases $f$, while decreasing $x$ while leaving $y$ and $z$ at zero, decreases $f$. 