# Homework Assignment #1

6.1 Suppose that the firm in Example 1 did not make any charitable contribution. Write out and solve the system of equations which describe its state and federal taxes. What is the net cost of its $5956 charitable contribution?

**Answer:** Profits are $100,000. State taxes are 5 percent of profits, or $5,000. Federal taxes are levied on profits minus state taxes and charitable contributions. Since there are no charitable contributions, federal taxes are 40 percent of $95,000, or $38,000. This means the firm earns $57,000 in after-tax profits.

In comparison, profits after taxes and contributions in Example 1 are $53,605. Thus after tax and contribution profits have fallen by $3,395 as a result of the $5,956 charitable contribution.

6.6 For the Markov employment model, Hall gives $p = .106$ and $q = .993$ for black females, and $p = .151$ and $q = .997$ for white females. Write out the Markov systems of difference equations for these two situations. Compute the stationary distributions.

**Answer:** For black females, the Markov system is:

$$\begin{aligned} x_{t+1} &= .993x_t + .106y_t \\ y_{t+1} &= .007x_t + .894y_t \end{aligned}$$

and for white females, the Markov system is:

$$\begin{aligned} x_{t+1} &= .997x_t + .151y_t \\ y_{t+1} &= .003x_t + .849y_t \end{aligned}$$

where $x_t$ and $y_t$ denote the average number employed and unemployed, respectively.

The stationary distribution is given by

$$x = \frac{p}{1 + p - 1} \qquad \text{and} \qquad y = \frac{1 - q}{1 + p - q}.$$

In the case of black females, this yields $x = .938$ and $y = .062$, while for white females, $x = .981$ and $y = .019$.

7.7 Use Gaussian elimination to solve

$$\begin{aligned} 3x + 3y &= 4 \\ -x - y &= 10 \end{aligned}$$

What happens and why?

**Answer:** We start by normalizing the top equation (divide it by 3).

$$\begin{aligned} x + y &= 4/3 \\ -x - y &= 10 \end{aligned}$$

Then we add the top equation to the bottom equation to eliminate the $x$ term:

We can see that there is no solution. The two original equations are inconsistent with each other. Had we converted it to matrix form, we would find the rank of the coefficient matrix is 1 while the rank of the augmented matrix is 2. Since the augmented matrix has higher rank, there is no solution.

7.22 The following five matrices from Exercise 7.20 are coefficient matrices of systems of linear equations. For each matrix, what can you say abou the number of solutions of the corresponding system: *a)* when the right-hand side is $b_1 = b_2 = b_m = 0$, and *b)* for general RHS $b_1, \ldots, b_m$?

$$a) \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \quad b) \begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \quad c) \begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{pmatrix},$$

$$d) \begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 1 & 9 & -6 & 4 & 9 \\ 1 & 3 & -8 & 4 & 2 \\ 2 & 15 & -13 & 11 & 16 \end{pmatrix}, \quad e) \begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{pmatrix}.$$

**Answer:** We begin by row-reducing each matrix and finding the rank.

*a)*

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \xrightarrow{(1)/2} \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \xrightarrow{(2)+(1)} \begin{pmatrix} 1 & -2 \\ 0 & -0 \end{pmatrix}.$$

Since the rank is 1 and there are two unknowns, the system has infinitely many solutions when $b_1 = b_2 = 0$. For general RHS, the system may not have a solution as $\text{rank}(A) < 2$.

*b)*

$$\begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix} \xrightarrow{(1)/2} \begin{pmatrix} 1 & -2 & 1 \\ -1 & 2 & 1 \end{pmatrix} \xrightarrow{(2)+(1)} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since the rank is 2 and there are three unknowns, the system has infinitely many solutions when $b_1 = b_2 = b_3 = 0$. For general RHS, the system will always have a solution as $\text{rank}(A) = 2$. As there are three variables (one free variable), there will always be infinitely many solutions.

*c)*

$$\begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{pmatrix} \xrightarrow{(2)-(1),(3)-(1)} \begin{pmatrix} 1 & 6 & -7 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & -3 & -1 & 1 \end{pmatrix} \xrightarrow{(3)+(2)} \begin{pmatrix} 1 & 6 & -7 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Since the rank is 3 and there are 4 unknowns, the system has infinitely many solutions when $b_1 = b_2 = b_3 = 0$. For general RHS, the system will always have a solution as $\text{rank}(A) = 3$. As there are four variables (one free variable), there will always be infinitely many solutions.

*d)*

$$\begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 1 & 9 & -6 & 4 & 9 \\ 1 & 3 & -8 & 4 & 2 \\ 2 & 15 & -13 & 11 & 16 \end{pmatrix} \xrightarrow{(2),(3)-(1),(4)-2(2)} \begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 0 & 3 & 1 & 1 & 4 \\ 0 & -3 & -1 & 1 & -3 \\ 0 & 3 & 1 & 5 & 6 \end{pmatrix} \xrightarrow{(3)+(2),(4)-(2)}$$

$$\begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 0 & 3 & 1 & 1 & 4 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 2 \end{pmatrix} \xrightarrow{(4)-2(3)} \begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 0 & 3 & 1 & 1 & 4 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the rank is 3 and there are 5 unknowns, the system has infinitely many solutions when $b_1 = b_2 = b_3 = b_4 = 0$. For general RHS, the fact that the rank is less than the number of rows means that the system will not have a solution for some RHS.

e)

$$\begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{pmatrix} \xrightarrow{(2),(3)-(1)} \begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & -3 & -1 & 1 & 4 \end{pmatrix} \xrightarrow{(3)+(2)} \begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{pmatrix}.$$

Since the rank is 3 and there are 5 unknowns, the system has infinitely many solutions when $b_1 = b_2 = b_3 = 0$. For general RHS, the system will always have a solution as rank$(A) = 3$. As there are five variables (two free variables), there will always be infinitely many solutions.

7.25 For each of the following two systems, we want to separate the variables into exogenous and endogenous ones so that each choice of values for the exogenous variables determines unique values for the endogenous variables. For each system *a)* determine how many variables can be endogenous at any one time, *b)* determine a successful separation into exogenous and endogenous variables, and *c)* find an explicit formula for the endogenous variables in terms of the exogenous ones:

$$\text{i)} \begin{aligned} x + 2y + z - w &= 1 \\ 3x + 6y - z - 3w &= 2; \end{aligned} \qquad \text{ii)} \begin{aligned} x + 2y + z - w &= 1 \\ 3x - y - 4z + 2w &= 3 \\ y + z + w &= 0. \end{aligned}$$

**Answer:** There are two equations and four variables in system (i), meaning that we can have at most two exogenous variables unless one of the equations is redundant. We row reduce system (i), obtaining

$$\begin{aligned} x + 2y - w &= 3/4 \\ z &= 1/4. \end{aligned}$$

It is clear that neither equation was redundant, and that $z$ must be an endogenous variable. We can group the variables by taking $x$ and $y$ as exogenous variables and $z$ and $w$ as the endogenous variables. The system above tells us that $z = 1/4$ and $w = -3/4 + x + 2y$.

We would also be successful by taking $(x, w)$ or $(y, w)$ as exogenous variables.

System (ii) has three equations and four variables, meaning that we can have at most one exogenous variables unless one of the equations is redundant. We row-reduce system (ii), obtaining

$$\begin{aligned} x - z &= 1 \\ y + z &= 0 \\ w &= 0. \end{aligned}$$

None of the equations were redundant, and $w$ must be endogenous. We can take $z$ as the exogenous variable and $x$, $y$, and $w$ as endogenous variables. Then $x = 1 + z$, $y = -z$ and $w = 0$. We would be equally successful taking $x$ or $y$ as the exogenous variable.