Homework Assignment #2

8.3 Show that if AB is defined, then $B^T A^T$ is defined but $A^T B^T$ need not be defined.

Answer: Suppose that A is $m \times k$ and B is $k' \times n$. The product AB will be defined iff $k = k'$, in which case AB is $m \times n$. Now B^T is $n \times k$ and A^T is $k \times m$, so $B^T A^T$ makes sense and is $n \times m$.

For $A^T B^T$ to make sense, we must have $m = n$. Thus $A = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ is an example of two matrices where AB makes sense, but $A^T B^T$ is not defined.

8.18 Show by simple matrix multiplication that, if $ad - bc \neq 0$,

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is both a left and right inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Answer:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = I_2$$

and

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = I_2.$$

8.29 Show that the inverse of a 2×2 symmetric matrix S is symmetric.

Answer: Let

$$S = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

be our symmetric matrix.

From problem 8.18, we know the inverse is

$$\frac{1}{ad - b^2} \begin{pmatrix} d & -b \\ -b & a \end{pmatrix}.$$

Of course, this requires $ad - b^2 \neq 0$. As is clear from it's form, the inverse matrix is also symmetric.

9.8 Use the observation following Theorem 9.2 to carry out a quick calculation of the determinant of each of the following matrices:

$$a) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{pmatrix} \qquad b) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 1 & 9 & 6 \end{pmatrix}.$$

Answer: We row reduce the first matrix by first subtracting the first row from the second and third rows, obtaining $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$, and then subtract the second row from the first, obtaining $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. We can find the determinant by multiplying the diagonal terms of this upper triangular matrix. The matrix in (a) has determinant 3.

For the second matrix, first subtract row 1 from row 3, and then subtract twice row 2 from row 3, obtaining $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & -5 \end{pmatrix}$. This has determinant -20.

NB: If you interchange any rows, don't forget to change the sign of the determinant each time.

9.13 Use Cramer's Rule to solve the following systems of equations:

$$\text{a)}\ \begin{aligned} 5x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 4 \end{aligned} \qquad \text{b)}\ \begin{aligned} 2x_1 - 3x_2 &= 2 \\ 4x_1 - 6x_2 + x_3 &= 7 \\ x_1 + 10x_2 &= 1. \end{aligned}$$

Answer: Cramer's Rule tells us that the solution to system (a) is

$$x_1 = \begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix} \Bigg/ \begin{vmatrix} 5 & 1 \\ 2 & -1 \end{vmatrix} = \frac{-7}{-7} = 1$$

$$x_2 = \begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} \Bigg/ \begin{vmatrix} 5 & 1 \\ 2 & -1 \end{vmatrix} = \frac{14}{-7} = -2.$$

The solution to system (b) is

$$x_1 = \begin{vmatrix} 2 & -3 & 0 \\ 7 & -6 & 1 \\ 1 & 10 & 0 \end{vmatrix} \Bigg/ \begin{vmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0 \end{vmatrix} = \frac{-23}{-23} = 1$$

$$x_2 = \begin{vmatrix} 2 & 2 & 0 \\ 4 & 7 & 1 \\ 1 & 1 & 0 \end{vmatrix} \Bigg/ \begin{vmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0 \end{vmatrix} = \frac{0}{-23} = 0$$

$$x_3 = \begin{vmatrix} 2 & -3 & 2 \\ 4 & -6 & 7 \\ 1 & 10 & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} 2 & -3 & 0 \\ 4 & -6 & 1 \\ 1 & 10 & 0 \end{vmatrix} = \frac{-69}{-23} = 3.$$

This can be easily checked by plugging the solutions back in the original equations.