Homework Assignment #3

12.15 Show that open balls on the real line are exactly the open intervals: sets of the form $(a, b) = \{x : a < x < b\}$, defined for two given numbers a and b.

Answer: We want to write $(a, b) = B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$. That requires $a = x - \varepsilon$ and $b = x + \varepsilon$. We solve for x and ε, obtaining

$$x = \frac{a + b}{2} \quad \text{and} \quad \varepsilon = \frac{b - a}{2} > 0.$$

12.16 Show that any open set is the union of open balls. Conclude that any open set is its own interior.

Answer: Let U be an open set in a metric space (X, d). Since U is open, for each $x \in U$, we can find an $\varepsilon_x > 0$ with $B_{\varepsilon_x}(x) \subset U$. Then $U = \cup_{x \in U} B_{\varepsilon_x}(x)$ since every $B_{\varepsilon_x}(x) \subset U$ and every point $x \in U$ is included in the union.

The interior is the union of all open sets contained in U. We've shown that U is the union of open balls contained in U, so the interior must at least contain U itself. Since $\operatorname{int} U \subset U$, it must be that $U = \operatorname{int} U$.

12.20 Prove that any finite set is a closed set. Prove that the set of integers is a closed set.

Answer: (1) Suppose A is a finite set.

(1a) The easy proof that A is closed applies in any metric space. In a metric space, any singleton is a closed set. Since any finite union of closed sets is closed, A is closed as the finite union of singletons.

(1b) For another proof that A is closed, let $\{x_n\}$ be a sequence in A with limit x. We must show that $x \in A$. We continue by contradiction. Suppose $x \notin A$ and let $\varepsilon = \min\{d(x, a) : a \in A\} > 0$. This minimum exists and is positive because A is finite and no point in A is x.

Then $d(x, a) \geq \varepsilon$ for all $a \in A$. But because $x_n \to x$, we can find and N with $d(x_n, x) < \varepsilon$ for $n > N$. This contradiction shows that $x \in A$. Since A contains all of its limit points, it is a closed set.

(2) Next we consider the status of the set of integers $\mathbb{Z}$. Let $\{x_n\}$ be a sequence in $\mathbb{Z}$ with limit x. We must show that $x \in \mathbb{Z}$. Since $x_n \to x$, there must be an N with $|x_n - x| < 1/2$ whenever $n \geq N$. Now take $k \geq N$. Then

$$|x_N - x_k| \leq |x_N - x| + |x - x_k| < 1/2 + 1/2 = 1.$$

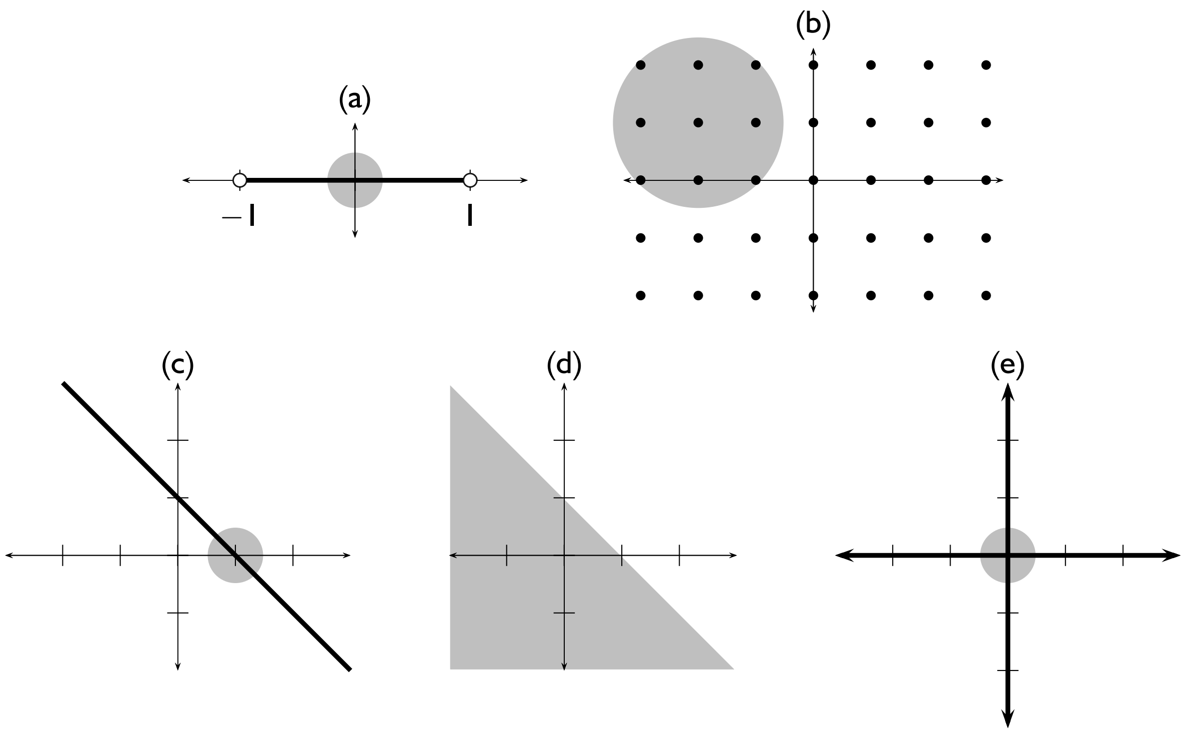
Since x_k and x_N are both integers, it must be that $x_k = x_N$ for all $k > N$. It follows that $\lim x_n = x_N \in \mathbb{Z}$. Since $\mathbb{Z}$ contains all of its limit points, it is a closed set.

12.21 For each of the following subsets of the plane, draw the set, state whether it is open, closed, or neither, and justify your answer in a word or two:

a) $\{(x, y) : -1 < x < +1, y = 0\}$, b) $\{(x, y) : x$ and y are integers$\}$,

c) $\{(x, y) : x + y = 1\}$, d) $\{(x, y) : x + y < 1\}$, e) $\{(x, y) : x = 0$ or $y = 0\}$.

Answer: First, we draw the sets.



a) This set, which I'll call A, is not open because $B_\varepsilon(0, 0)$ pokes out of A as illustrated, nor is A closed as the limit points $(\pm 1, 0)$ are not in A.

b) This set, which I'll call B, is not open. For $\varepsilon > 0$, any ε-ball about any point in the set contains points not in the set (illustrated). This set is closed. As in problem 12.20, any convergent sequence with integral coordinates must eventually be constant since there is only one integer point within any distance $\varepsilon < 1/2$ of its limit.

c) This set is not open ($B_\varepsilon(1, 0)$ pokes out as illustrated). It is closed. To see this, define $f(x, y) = x + y$, then f is continuous and the set is $f^{-1}(\{1\})$, which is closed as the inverse image of a closed set.

d) This set, which I'll call D, is open. Use the function f from (c), so that $D = f^{-1}(-\infty, 0))$. It is not closed since $x_n = (1, -1/n) \in D$ and $x_n \to (1, 0) \notin D$.

e) This set, which I will call E, is not open as $B_\varepsilon(0,0) \not\subset E$ as illustrated. To see that E is closed, consider the function $f(x,y) = xy$. As a polynomial, this is a continuous function. The set is the inverse image of the singleton $\{0\}$, $E = f^{-1}(\{0\})$. Since singletons are closed, and f is continuous, E is also closed.

13.17 Suppose that $f\colon \mathbb{R}^k \to \mathbb{R}^1$ is a continuous function and that $f(\mathbf{x}^*) > 0$. Show that there is a ball $B = B_\delta(\mathbf{x}^*)$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in B$.

Answer: The set $\mathbb{R}_{++} = (0, +\infty)$ is an open set. Since f is continuous, $f^{-1}(\mathbb{R}_{++})$ is an open set. Moreover, $\mathbf{x}^* \in f^{-1}(\mathbb{R}_{++})$. Since $\mathbf{x}^*$ is a point in an open set, we can put an ball around it that is in that open set. We can choose $\delta > 0$ so that $B_\delta(\mathbf{x}^*) \subset f^{-1}(\mathbb{R}_{++})$. Then $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in B_\delta(\mathbf{x}^*)$.