Homework Assignment #4

29.3 Prove that if a sequence converges, every subsequence of it converges too.

Answer: Suppose $\{x_n\}$ is a sequence of real numbers with $x_n \to x$ and let x_{n_k} be a subsequence. Take any $\varepsilon > 0$. Since $x_n \to x$, there is a $N > 0$ with $|x_n - x| < \varepsilon$ whenever $n \geq N$. Since $\{x_{n_k}\}$ is a subsequence, there is a $K > 0$ with $n_k \geq N$ whenever $k \geq K$. Then $|x_{n_k} - x| < \varepsilon$ whenever $k \geq K$, showing that $x_{n_k} \to x$.

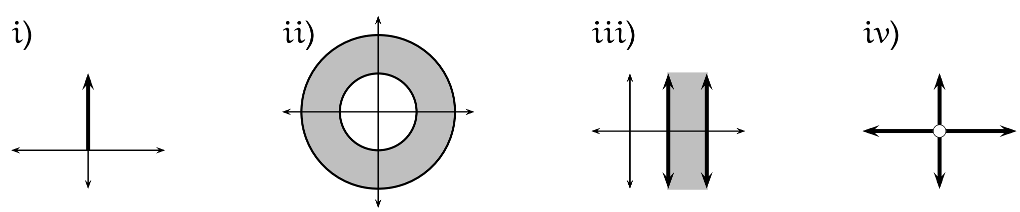
29.9 Give an example to show that the interior of a connected set need not be connected.

Answer: There are many possible examples. Here is one. Let $S = \{(x, y) : |y| \leq |x|\}$. It is a connected set (in fact, it is star-shaped relative to the origin). Its interior is $\operatorname{int} S = \{(x, y) : |y| < |x|\}$, which fails to be connected because the origin is not in $\operatorname{int} S$.

29.11 For each of the following subsets of $\mathbb{R}^2$, a) sketch the set and b) determine whether or not it is open, closed, compact, or connected. Give reasons for your negative answers to part b.

$$i)\ \{(x, y) : x = 0, y \geq 0\}, \qquad ii)\ \{(x, y) : 1 \leq x^2 + y^2 \leq 2\},$$
$$iii)\ \{(x, y) : 1 \leq x \leq 2\}, \qquad iv)\ \{(x, y) : x = 0 \text{ or } y = 0, \text{but not both}\}.$$

Answer: We start with the illustrations.



i) The set is closed and connected. Any ball around $(0, 0)$ contains points outside the set, so it is not open. It is not compact because is it not bounded (any point $(0, y)$ for $y \geq 0$ is in the set).

ii) The set is closed, compact, and connected. It is not open because no open ball about $(1, 0)$ is contained in the set.

iii) The set is closed and connected. It is not open because no ball around $(1, 0)$ is contained in the set. It is not compact because it is not bounded.

iv) The set is *not* open, closed, compact, or connected. It is not open because it contains no open ball about the point $(1, 0)$. It is not closed because the limit point $(0, 0) = \lim(1/n, 0)$ is not in the set. It is not compact because it is not closed. Finally, it is not connected because the open sets $U = \{(x, y) : x + y > 0\}$ and $V = \{(x, y) : x + y < 0\}$ disconnect it.

29.13 Show that $N_{(a_1,\dots,a_n)}$ is a norm on $\mathbb{R}^n$ where $N_{(a_1,\dots,a_n)}(x) = \|(a_1^{1/2}x_1, \dots, a_n^{1/2}x_n)\|_2$ is the weighted Euclidean norm on $\mathbb{R}^n$.

Answer: Here we assume each $a_i > 0$, as in the book. By definition,

$$N^2_{(a_1,\dots,a_n)}(x) = \left\|(a_1^{1/2}x_1, \dots, a_n^{1/2}x_n)\right\|_2.$$

It follows that $N^2_{(a_1,\dots,a_n)}(x) \geq 0$ and that $N^2_{(a_1,\dots,a_n)}(x) = 0$ if and only if $(a_1^{1/2}x_1, \dots, a_n^{1/2}x_n) = 0$. The latter happens if and only if each $a_i^{1/2}x_i = 0$. Since each $a_i > 0$, that is equivalent to $x = 0$. This establishes that N is positive definite.

Now

$$\begin{aligned} N^2_{(a_1,\dots,a_n)}(\alpha x) &= \left\|\alpha(a_1^{1/2}x_1, \dots, a_n^{1/2}x_n)\right\|_2 \\ &= |\alpha| \left\|(a_1^{1/2}x_1, \dots, a_n^{1/2}x_n)\right\|_2 \\ &= |\alpha| \, N^2_{(a_1,\dots,a_n)}(x). \end{aligned}$$

This shows $N^2_{(a_1,\dots,a_n)}(x)$ is absolutely homogeneous of degree one.

Finally,

$$\begin{aligned} N^2_{(a_1,\dots,a_n)}(x+y) &= \left\|(a_1^{1/2}(x_1+y_1), \dots, a_n^{1/2}(x_n+y_n))\right\|_2 \\ &\leq \left\|(a_1^{1/2}x_1, \dots, a_n^{1/2}x_n)\right\|_2 + \left\|(a_1^{1/2}y_1, \dots, a_n^{1/2}y_n)\right\|_2 \\ &= N^2_{(a_1,\dots,a_n)}(x) + N^2_{(a_1,\dots,a_n)}(y) \end{aligned}$$

where we used the ordinary triangle inequality for $\|\cdot\|$. This establishes the triangle inequality.