14.2 Compute the partial derivatives of the Cobb-Douglas production function \( q = k x_1^{a_1} x_2^{a_2} \) and of the Constant Elasticity of Substitution production function \( q = k (c_1 x_1^{-a} + c_2 x_2^{-a})^{-h/a} \), assuming all the parameters are positive.

**Answer:** For the Cobb-Douglas function,

\[
\frac{\partial q}{\partial x_1} = k a_1 x_1^{a_1-1} x_2^{a_2} = \frac{a_1 q}{x_1}
\]

and

\[
\frac{\partial q}{\partial x_2} = k a_2 x_1^{a_1} x_2^{a_2-1} = \frac{a_2 q}{x_2}.
\]

For the CES production function,

\[
\frac{\partial q}{\partial x_1} = h k c_1 x_1^{-1-a} (c_1 x_1^{-a} + c_2 x_2^{-a})^{-1-h/a}
\]

and

\[
\frac{\partial q}{\partial x_2} = h k c_2 x_2^{-1-a} (c_1 x_1^{-a} + c_2 x_2^{-a})^{-1-h/a}.
\]

14.4 Consider the production function \( Q = 9L^{2/3}K^{1/3} \).

a) What is the output when \( L = 1000 \) and \( K = 216 \)?

**Answer:** Here \( 1000 = 10^3 \) and \( 216 = 6^3 \), so \( Q = 9(100)(6) = 5400 \).

b) Use marginal analysis to estimate \( Q(998, 216) \) and \( Q(1000, 217.5) \).

**Answer:** Now \( Q(998, 216) \approx 5400 - 2(\partial Q/\partial L) \) and \( Q(1000, 217.5) \approx 5400 + 1.5(\partial Q/\partial K) \). Here \( (\partial Q/\partial L)(1000, 216) = 3.6 \) and \( (\partial Q/\partial K)(1000, 216) = 25/3 \), so the approximate values are \( Q(998, 216) \approx 5392.8 \) and \( Q(1000, 217.5) \approx 5412.5 \).

c) Use a calculator to compute these two values of \( Q \) to three decimal places and compare these values with your estimates in b.

**Answer:** According to the calculator, \( Q(998, 216) \approx 5392.798 \), a discrepancy of 0.002 and \( Q(1000, 217.5) \approx 5412.471 \), a discrepancy of 0.029.

d) How big must \( \Delta L \) be in order for the difference between \( Q(1000 + \Delta L, 216) \) and its linear approximation, \( Q(1000, 216) + (\partial Q/\partial L)(1000, 216)\Delta L \), to differ by more than 2 units? (Plug increasing values of \( L \) into these two expressions.)
14.8 Use differentials to approximate each of the following:

a) \( f(x, y) = x^4 + 2x^2y^2 + xy^4 + 10y \) at \( x = 10.36 \) and \( y = 1.04 \);

**Answer:** \( df = (4x^3 + 4xy^2 + y^4)\, dx + (4x^2y + 4xy^3 + 10)\, dy \). We evaluate this at the point \((10, 1)\), obtaining \( df = 4041\, dx + 450\, dy \). Replacing \( dx \) by \( \Delta x = .36 \) and \( dy \) by \( \Delta y = .04 \), we find \( \Delta f \approx 4041(.36) + 450(.04) = 1454.76 + 18 = 1472.76 \). Since \( f(10, 1) = 10220 \), the approximate value of the function is 11692.76 while the actual value is about 11744.3.

b) \( f(x, y) = 6x^{2/3}y^{1/2} \) at \( x = 998 \) and \( y = 101.5 \);

**Answer:** Here we approximate around \((x, y) = (1000, 100)\), where \( f(1000, 100) = 6000 \). Then \( df = 4x^{-1/3}y^{1/2}\, dx + 3x^{2/3}y^{-1/2}\, dy \). Evaluating at \((1000, 100)\), we find \( df = 4\, dx + 30\, dy \). Replacing \( dx \) by \( \Delta x = -2 \) and \( dy \) by \( \Delta y = 1.5 \), we find \( \Delta f \approx -8 + 45 = 37 \), so \( f(998, 101.5) \approx 6037 \). In comparison, the actual value is about 6036.77.

c) \( f(x, y, z) = \sqrt{x^{1/2} + y^{1/3} + 5z^2} \) at \( x = 4.2 \), \( y = 7.95 \), and \( z = 1.02 \).

**Answer:** Here we evaluate around the point \((x, y, z) = (4, 8, 1)\). Note that \( f(4, 8, 1) = 3 \). Then

\[
df = \frac{1}{2\sqrt{x^{1/2} + y^{1/3} + 5z^2}} \left[ \frac{1}{2} x^{-1/2} \, dx + \frac{1}{3} y^{-2/3} \, dy + 10z \, dz \right]
\]

At \((4, 8, 1)\), this becomes \( \frac{1}{6} [\frac{1}{4} \, dx + \frac{1}{12} \, dy + 10 \, dz] \). We use \( \Delta x = 0.2 \), \( \Delta y = -0.05 \), and \( \Delta z = 0.02 \). This yields \( \Delta f \approx 0.04097 \), so \( f(4.2, 7.95, 1.02) \approx 3.04097 \), compared to the actual value of about 3.04092.

14.28 The goal of this exercise is to examine a \( C^1 \) function for which the conclusion of Theorem 14.5 fails—the cross partials are not equal. Let

\[
f(x, y) = \begin{cases} 
0 & \text{if } (x, y) = (0, 0), \\
\frac{x^3y - xy^3}{x^2 + y^2} & \text{otherwise.}
\end{cases}
\]

a) Prove that \( f \) is zero along the \( x \)-axis and the \( y \)-axis. Conclude that \( (\partial f/\partial x)(0, 0) \) and \( (\partial f/\partial y)(0, 0) \) are both 0.

b) Compute \( \partial f/\partial x \) and \( \partial f/\partial y \) for \((x, y) \neq (0, 0)\).
c) Conclude that \((\partial f/\partial x)(0, y) = -y\) and \((\partial f/\partial x)(x, 0) = x\).

d) Show that

\[
\frac{\partial^2 f}{\partial y \partial x}(0, 0) = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)(0, 0) = \lim_{y \to 0} \frac{\frac{\partial f}{\partial x}(0, y) - \frac{\partial f}{\partial x}(0, 0)}{y} = -1.
\]

e) Show that

\[
\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)(0, 0) = \lim_{x \to 0} \frac{\frac{\partial f}{\partial y}(x, 0) - \frac{\partial f}{\partial y}(0, 0)}{x} = +1
\]

and conclude that the mixed partials are not equal at \((0, 0)\).

f) Compute \((\partial^2 f/\partial x \partial y)(x, y)\) for all \((x, y)\) other than \((0, 0)\).

\[
\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)(x, y) = \frac{x^5 - 4x^3y^2 + xy^4}{(x^2 + y^2)^2}.
\]

g) Use \(f\) to show that \((\partial^2 f/\partial x \partial y)(x, x) = 0\) for \(x > 0\).

h) Compare \(e\) and \(g\) to show that \((\partial^2 f/\partial x \partial y)(x, y)\) is discontinuous at the origin. Therefore, \(f\) is not \(C^2\) and the hypotheses of Theorem 14.5 do not hold.

**Answer:**

a) For \(y \neq 0\), \(f(0, y) = 0/y^2 = 0\) and for \(x \neq 0\), \(f(x, 0) = 0/x^2 = 0\). Finally, \(f(0, 0) = 0\) by definition.

b) Now

\[
\frac{\partial f}{\partial x} = \frac{3x^2y - y^3}{x^2 + y^2} - 2x \frac{x^2y - xy^3}{(x^2 + y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}
\]

and

\[
\frac{\partial f}{\partial y} = \frac{x^3 - 3xy^2}{x^2 + y^2} - 2y \frac{x^2y - xy^3}{(x^2 + y^2)^2} = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}.
\]

c) For \(y \neq 0\), \((\partial f/\partial x)(0, y) = -y^5/y^4 = -y\). For \(x \neq 0\), \((\partial f/\partial y)(x, 0) = x^5/x^4 = x\).

d) The expression is just the difference quotient defining \(\partial^2 f/\partial x \partial y\). It is true because

\[
\frac{\frac{\partial f}{\partial y}(0, y) - \frac{\partial f}{\partial y}(0, 0)}{y} = \frac{-y}{y} = -1.
\]

e) The expression is just the difference quotient defining \(\partial^2 f/\partial x \partial y\). It is true because

\[
\frac{\frac{\partial f}{\partial y}(x, 0) - \frac{\partial f}{\partial y}(0, 0)}{x} = \frac{x}{x} = +1.
\]

The order of differentiation makes a difference here.
f) Now

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{x^8 + 10x^6y^2 - 10x^2y^6 - y^8}{(x^2 + y^2)^4} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3}.
\]

g) Setting \( y = x \neq 0 \), we find

\[
\frac{\partial^2 f}{\partial x \partial y} = 0 = \frac{0}{2^8x^8} = 0.
\]

h) Now

\[
\lim_{{x \to 0}} \frac{\partial^2 f}{\partial x \partial y}(x, x) = 0 \neq +1 = \frac{\partial^2 f}{\partial x \partial y}(0, 0)
\]

so the second derivative is not continuous at \((0, 0)\).