Homework Assignment #5

14.2 Compute the partial derivatives of the Cobb-Douglas production function $q = kx_1^{a_1}x_2^{a_2}$ and of the Constant Elasticity of Substitution production function $q = k(c_1x_1^{-a} + c_2x_2^{-a})^{-h/a}$, assuming all the parameters are positive.

Answer: For the Cobb-Douglas function,

$$\frac{\partial q}{\partial x_1} = ka_1x_1^{a_1-1}x_2^{a_2} = \frac{a_1q}{x_1}$$

and

$$\frac{\partial q}{\partial x_2} = ka_2x_1^{a_1}x_2^{a_2-1} = \frac{a_2q}{x_2}.$$

For the CES production function,

$$\frac{\partial q}{\partial x_1} = hkc_1x_1^{-1-a}(c_1x_1^{-a} + c_2x_2^{-a})^{-1-h/a}$$

and

$$\frac{\partial q}{\partial x_2} = hkc_2x_2^{-1-a}(c_1x_1^{-a} + c_2x_2^{-a})^{-1-h/a}.$$

14.4 Consider the production function $Q = 9L^{2/3}K^{1/3}$.

a) What is the output when $L = 1000$ and $K = 216$?

Answer: Here $1000 = 10^3$ and $216 = 6^3$, so $Q = 9(100)(6) = 5400$.

b) Use marginal analysis to estimate $Q(998, 216)$ and $Q(1000, 217.5)$.

Answer: Now $Q(998, 216) \approx 5400-2(\partial Q/\partial L)$ and $Q(1000, 217.5) \approx 5400+1.5(\partial Q/\partial K)$. Here $(\partial Q/\partial L)(1000, 216) = 3.6$ and $(\partial Q/\partial K)(1000, 216) = 25/3$, so the approximate values are $Q(998, 216) \approx 5392.8$ and $Q(1000, 217.5) \approx 5412.5$.

c) Use a calculator to compute these two values of Q to three decimal places and compare these values with your estimates in *b*.

Answer: According to the calculator, $Q(998, 216) \approx 5392.798$, a discrepancy of 0.002 and $Q(1000, 217.5) \approx 5412.471$, a discrepancy of 0.029.

d) How big must ΔL be in order for the difference between $Q(1000 + \Delta L, 216)$ and its linear approximation, $Q(1000, 216) + (\partial Q/\partial L)(1000, 216)\Delta L$, to differ by more than 2 units? (Plug increasing values of L into these two expressions.)

Answer: Here $Q(1000, 216) + (\partial Q/\partial L)(1000, 216)\Delta L = 5400 + 3.6\Delta L$ and $Q(1000 + \Delta L, 216) = 54(1000 + \Delta L)^{2/3}$. Using a spreadsheet, I found that at $\Delta L = 58.48$ the difference is just over 2 units (2.0004). The same was true at $\Delta L = -57$, where the difference was 2.0005.

14.8 Use differentials to approximate each of the following:

a) $f(x, y) = x^4 + 2x^2y^2 + xy^4 + 10y$ at $x = 10.36$ and $y = 1.04$;

Answer: $df = (4x^3 + 4xy^2 + y^4)\,dx + (4x^2y + 4xy^3 + 10)\,dy$. We evaluate this at the point (10, 1), obtaining $df = 4041\,dx + 450\,dy$. Replacing dx by $\Delta x = .36$ and dy by $\Delta y = .04$, we find $\Delta f \approx 4041(.36) + 450(.04) = 1454.76 + 18 = 1472.76$. Since $f(10, 1) = 10220$, the approximate value of the function is 11692.76 while the actual value is about 11744.3.

b) $f(x, y) = 6x^{2/3}y^{1/2}$ at $x = 998$ and $y = 101.5$;

Answer: Here we approximate around $(x, y) = (1000, 100)$, where $f(1000, 100) = 6000$. Then $df = 4x^{-1/3}y^{1/2}\,dx + 3x^{2/3}y^{-1/2}\,dy$. Evaluating at (1000, 100), we find $df = 4\,dx + 30\,dy$. Replacing dx by $\Delta x = -2$ and dy by $\Delta y = 1.5$, we find $\Delta f \approx -8 + 45 = 37$, so $f(998, 101.5) \approx 6037$. In comparison, the actual value is about 6036.77.

c) $f(x, y, z) = \sqrt{x^{1/2} + y^{1/3} + 5z^2}$ at $x = 4.2$, $y = 7.95$, and $z = 1.02$.

Answer: Here we evaluate around the point $(x, y, z) = (4, 8, 1)$. Note that $f(4, 8, 1) = 3$. Then

$$df = \frac{1}{2\sqrt{x^{1/2} + y^{1/3} + 5z^2}} \left[\frac{1}{2}x^{-1/2}\,dx + \frac{1}{3}y^{-2/3}\,dy + 10z\,dz\right].$$

At (4, 8, 1), this becomes $\frac{1}{6}[\frac{1}{4}\,dx + \frac{1}{12}\,dy + 10\,dz]$. We use $\Delta x = 0.2$, $\Delta y = -0.05$, and $\Delta z = 0.02$. This yields $\Delta f \approx 0.04097$, so $f(4.2, 7.95, 1.02) \approx 3.04097$, compared to the actual value of about 3.04092.

14.28 The goal of this exercise is to examine a $\mathcal{C}^1$ function for which the conclusion of Theorem 14.5 fails—the cross partials are not equal. Let

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \dfrac{x^3y - xy^3}{x^2 + y^2} & \text{otherwise.} \end{cases}$$

a) Prove that f is zero along the x-axis and the y-axis. Conclude that $(\partial f/\partial x)(0, 0)$ and $(\partial f/\partial y)(0, 0)$ are both 0.

b) Compute $\partial f/\partial x$ and $\partial f/\partial y$ for $(x, y) \neq (0, 0)$.

c) Conclude that $(\partial f/\partial x)(0, y) = -y$ and $(\partial f/\partial x)(x, 0) = x$.

d) Show that

$$\frac{\partial^2 f}{\partial y\, \partial x}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{y\to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y} = -1.$$

e) Show that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{x\to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x} = +1$$

and conclude that the mixed partials are not equal at $(0, 0)$.

f) Compute $(\partial^2 f/\partial x\partial y)(x, y)$ for all (x, y) other than $(0, 0)$.

g) Use f to show that $(\partial^2 f/\partial x\partial y)(x, x) = 0$ for $x > 0$.

h) Compare e and g to show that $(\partial^2 f/\partial x\partial y)(x, y)$ is discontinuous at the origin. Therefore, f is not $\mathcal{C}^2$ and the hypotheses of Theorem 14.5 do not hold.

Answer:

a) For $y \neq 0$, $f(0, y) = 0/y^2 = 0$ and for $x \neq 0$, $f(x, 0) = 0/x^2 = 0$. Finally, $f(0, 0) = 0$ by definition.

b) Now

$$\frac{\partial f}{\partial x} = \frac{3x^2y - y^3}{x^2 + y^2} - 2x\frac{x^3y - xy^3}{(x^2 + y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{x^3 - 3xy^2}{x^2 + y^2} - 2y\frac{x^3y - xy^3}{(x^2 + y^2)^2} = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}.$$

c) For $y \neq 0$, $(\partial f/\partial x)(0, y) = -y^5/y^4 = -y$. For $x \neq 0$, $(\frac{\partial f}{\partial y})(x, 0) = x^5/x^4 = x$.

d) The expression is just the difference quotient defining $\partial^2 f/\partial x\partial y$. It is true because

$$\frac{\frac{\partial f}{\partial y}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y} = \frac{-y}{y} = -1.$$

e) The expression is just the difference quotient defining $\partial^2 f/\partial x\partial y$. It is true because

$$\frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x} = \frac{x}{x} = +1.$$

The order of differentiation makes a difference here.

f) Now

$$\begin{aligned}\frac{\partial^2 f}{\partial x\,\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\\ &= \frac{x^8 + 10x^6y^2 - 10x^2y^6 - y^8}{(x^2+y^2)^4}\\ &= \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}.\end{aligned}$$

g) Setting $y = x \neq 0$, we find

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{0}{2^8x^8} = 0.$$

h) Now

$$\lim_{x\to 0}\frac{\partial^2 f}{\partial x\,\partial y}(x,x) = 0 \neq +1 = \frac{\partial^2 f}{\partial x\,\partial y}(0,0)$$

so the second derivative is not continuous at $(0,0)$.