Homework Assignment #6

15.17 Consider the system of equations

$$y^2 + 2u^2 + v^2 - xy = 15, \qquad 2y^2 + u^2 + v^2 + xy = 38,$$

at the solution $x = 1$, $y = 4$, $u = 1$, and $v = -1$. Think of u and v as exogenous and x and y as endogenous. Use calculus to estimate the values of x and y that correspond to $u = 0.9$ and $v = -1.1$.

Answer: Rather than just "using calculus", I've made of the Implicit Function Theorem to organize everything. Calculus is still needed, but has an IFT wrapper around it.

Let

$$g = \begin{pmatrix} y^2 + 2u^2 + v^2 - xy \\ 2y^2 + u^2 + v^2 + xy \end{pmatrix}.$$

Then

$$Dg = \begin{pmatrix} -y & 2y - x & 4u & 2v \\ y & 4y + x & 2u & 2v \end{pmatrix}.$$

Evaluating at $x^* = (1, 4, 1, -1)$, we obtain

$$Dg(x^*) = \begin{pmatrix} -4 & 7 & 4 & -2 \\ 4 & 17 & 2 & -2 \end{pmatrix}.$$

To estimate the values, we need to calculate $\left[D_{(x,y)}(u, v)\right](x*)$. By the Implicit Function Theorem, we first calculate the derivative of the first two columns of g and find out if it's invertible at $x^* = (1, 4, 1, -1)$. Here

$$D_{(x,y)}g(x^*) = \begin{pmatrix} -4 & 7 \\ 4 & 17 \end{pmatrix} \quad \text{and its inverse is} \quad \frac{-1}{96}\begin{pmatrix} 17 & -7 \\ -4 & -4 \end{pmatrix}.$$

Then

$$\begin{aligned} [D_{(x,y)}(u, v)](x^*) &= -\left[D_{(x,y)}g(x^*)\right]^{-1} D_{(u,v)}g(x^*) \\ &= \frac{1}{96}\begin{pmatrix} 17 & -7 \\ -4 & -4 \end{pmatrix} \times \begin{pmatrix} 4 & -2 \\ 2 & -2 \end{pmatrix} \\ &= \frac{1}{48}\begin{pmatrix} 27 & -10 \\ -12 & 8 \end{pmatrix} = \begin{pmatrix} 9/16 & -5/24 \\ -1/4 & +1/6 \end{pmatrix}. \end{aligned}$$

Setting $(\Delta u, \Delta v) = (-0.1, -0.1)$, we find

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \frac{1}{48}\begin{pmatrix} 27 & -10 \\ -12 & 8 \end{pmatrix} \times \begin{pmatrix} -0.1 \\ -0.1 \end{pmatrix} = \frac{1}{48}\begin{pmatrix} -1.7 \\ 0.4 \end{pmatrix} = \begin{pmatrix} -0.0354 \\ 0.00833 \end{pmatrix}$$

The estimated values are $x = 0.9646$ and $y = 4.00833$.

15.38 Show that $F(x, y) = (e^y \cos x, e^y \sin x)$ is locally one-to-one and onto, but not globally one-to-one.

Answer: We start by stating the definitions not covered in class (S&B pg. 365).

Let $\mathbf{x}_0$ be a point in the domain of $F\colon \mathbb{R}^n \to \mathbb{R}^n$ with $F(\mathbf{x}_0) = \mathbf{b}_0$.

Then F is **locally onto** at $\mathbf{x}_0$ if, given any open ball $B_r(\mathbf{x}_0) \subset \mathbb{R}^n$, there is a ball $B_s(\mathbf{b}_0) \subset \mathbb{R}^m$ such that for every $\mathbf{b} \in B_s(\mathbf{b}_0)$ there is at least one $\mathbf{x} \in B_r(\mathbf{x}_0)$ with $F(\mathbf{x}) = \mathbf{b}$.

Similarly, F as above is **locally one-to-one** at $\mathbf{x}_0$ if there is a ball $B_r(\mathbf{x}_0)$ and a ball $B_s(\mathbf{b}_0)$ such that for every $\mathbf{b} \in B_s(\mathbf{b}_0)$ there is at most one $\mathbf{x} \in B_r(\mathbf{x}_0)$ such that $F(\mathbf{x}) = \mathbf{b}$.

The fact that F is both locally onto and locally one-to-one follows from the inverse function theorem (see the discussion on S&B pp. 365–367). Here the Jacobian of F is

$$DF = \begin{pmatrix} -e^y \sin x & e^y \cos x \\ e^y \cos x & e^y \sin x \end{pmatrix}.$$

As this has determinant $-e^{2y} \sin^2 x - e^{2y} \cos^2 x = -e^{2y} \neq 0$, the Inverse Function Theorem applies.

That leaves the global question. For all $(x, y) \in \mathbb{R}^2$ and integers n, $F(x, y) = F(x + 2n\pi, y)$, so F is not globally one-to-one.

16.1 Determine the definiteness of the following symmetric matrices.

$$\text{a)} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{b)} \begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix} \quad \text{c)} \begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix} \quad \text{d)} \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$$

$$\text{e)} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix} \quad \text{f)} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{g)} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$$

Answer: In each case, we start by computing the leading principal minors. If necessary, we will compute all principal minors.

a) **Positive Definite** because $A_1 = 2 > 0$ and $A_2 = 1 > 0$.

b) **Indefinite** because $A_1 = -3 < 0$ and $A_2 = -1 < 0$.

c) **Negative Definite** because $A_1 = -3 < 0$ and $A_2 = 2 > 0$.

d) **Positive Semidefinite** because $A_1 = 2 > 0$, $A_2 = 0$, and $a_{22} = 8 > 0$.

e) **Indefinite** because $A_1 = 1 > 0$, $A_2 = 0$, $A_3 = -25 < 0$.

f) **Negative Semidefinite** because $A_1 = -1 < 0$, $A_2 = 0$, $A_3 = 0$, $a_{22} = -4 < 0$ $a_{33} = -2 < 0$, and the other two second order minors are both $2 > 0$.

g) **Indefinite** because $A_1 = 1 > 0$, $A_2 = 2 > 0$, $A_3 = -10 < 0$ and $A_4 = 65 > 0$.

16.6 Determine the definiteness of the following constrained quadratics.

a) $Q(x_1, x_2) = x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.

b) $Q(x_1, x_2) = 4x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.

c) $Q(x_1, x_2, x_3) = x_1^2+x_2^2-x_3^2+4x_1x_3-2x_1x_2$, subject to $x_1+x_2+x_3 = 0$ and $x_1+x_2-x_3 = 0$.

d) $Q(x_1, x_2, x_3) = x_1^2+x_2^2+x_3^2+4x_1x_3-2x_1x_2$, subject to $x_1+x_2+x_3 = 0$ and $x_1+x_2-x_3 = 0$.

e) $Q(x_1, x_2, x_3) = x_1^2 - x_3^2 + 4x_1x_2 - 6x_2x_3$, subject to $x_1 + x_2 - x_3 = 0$.

Answer:

a) Here $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and the bordered Hessian is

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

There are $n = 2$ variables and $m = 1$ constraints, so we must look at the last leading principal minor, H_3. We find $H_3 = 2$. It has the same sign as $(-1)^n = 1$, so the constrained quadratic is **negative definite**.

b) Set $A = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$. The bordered Hessian is

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Again, $n = 2$ and $m = 1$, so we must look at the last leading principal minor. Then $H_3 = -1$ which has the same sign as $(-1)^m = -1$, so it is **positive definite**.

c) Set $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}$. The bordered Hessian is

$$H = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & -1 \end{pmatrix}.$$

Now there are $n = 3$ variables and $m = 2$ constraints, so we look at the last leading principal minor. Then $H_5 = 16$ which has the same sign as $(-1)^m = +1$, so it is **positive definite**.

d) Set $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$. The bordered Hessian is

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & 1 \end{pmatrix}.$$

As in (c), there are $n = 3$ variables and $m = 2$ constraints. We again check the last leading principal minor. We find $H_5 = 16$, which has the same sign as $(-1)^m = +1$. It is **positive definite**.

e) Set $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -3 \\ 0 & -3 & -1 \end{pmatrix}$. The bordered Hessian is

$$\begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & -3 \\ -1 & 0 & -3 & -1 \end{pmatrix}.$$

There are $n = 3$ variables and $m = 1$ constraints, so we check the last 2 leading principal minors. We obtain $H_3 = 3$ and $H_4 = 4$. Both have the same sign, but the sign of H_4 is neither $(-1)^m = -1$ or $(-1)^n = -1$, and the form is **indefinite** on the constraint set.

17.9

a) Prove that $2ab \le a^2 + b^2$ for all numbers a, b.

Answer: Here $0 \le (a-b)^2 = a^2 - 2ab + b^2$. Rearrange to complete the proof.

b) Use this result to show that

$$\begin{aligned}(x_1 + \cdots + x_n)^2 &= x_1^2 + \cdots + x_n^2 + \sum_{i<j} 2x_i x_j \\ &\le x_1^2 + \cdots + x_n^2 + (n-1)(x_1^2 + \cdots + x_n^2) \\ &= n(x_1^2 + \cdots + x_n^2)\end{aligned}$$

Answer: The case $n = 1$ is trivial. Now suppose $n > 1$. $2x_i x_j \le (x_i^2 + x_j^2)$. When we look at the sum $\sum_{i<j} 2x_i x_j \le \sum_{i<j} x_i^2 + x_j^2$, we find that each x_k appears $(n-1)$ times, so $\sum_{i<j} x_i^2 + x_j^2 = (n-1)\sum_{i=1}^n x_i^2$, yielding line 2. Line follows immediately.

This inequality can also be written as

$$\left(\sum_{i=1}^n x_i\right)^2 \le n\left(\sum_{i=1} x_i^2\right).$$

c) Conclude that the point (m^*, b^*) in (14) and (15) is a global minimizer of the function S in (11).

Answer: A sufficient condition for a global maximum is that the Hessian is everywhere positive semidefinite. This is equivalent to all principal minors being non-negative. It is not enough to look at the leading principal minors.

The Hessian of S is

$$H = \begin{pmatrix} 2\sum_i x_i^2 & 2\sum_i x_i \\ 2\sum_i x_i & 2n \end{pmatrix}.$$

The two first order principal minors are $2\sum_i x_i^2 \ge 0$ (the leading first order principal minor) and $2n \ge 0$ (the other first order principal minor).

There's only one second order principal minor, the determinant of H. We calculate

$$\det H = 4\left[n\left(\sum_i x_i^2\right) - \left(\sum_i x_i\right)^2\right].$$

By part (b), $\det H \ge 0$.

All three principal minors are of H are positive semidefinite for all $(m, b) \in \mathbb{R}^2$. This shows the solution to the first order conditions is a global minimum.