Homework Assignment #7

18.2 Find the maximum and minimum distance from the origin to the ellipse $x^2 + xy + y^2 = 3$. [Hint: Use $x^2 + y^2$ as your objective function.]

Answer: The problem is to maximize (minimize) $x^2 + y^2$ subject to the constraint $x^2 + xy + y^2 = 3$. Note that the constraint function has derivative $dh = (2x + y, x + 2y)$ which is non-zero on the ellipse $x^2 + xy + y^2 = 3$. This establishes constraint qualification.

We then form the Lagrangian $\mathcal{L} = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 3)$, which yields first-order conditions

$$0 = 2x - \lambda(2x + y)$$
$$0 = 2y - \lambda(x + 2y).$$

We divide to eliminate λ, obtaining $x/y = (2x + y)/(2y + x)$. Clearing the fractions yields $x^2 = y^2$.

There are two cases: $x = y$ and $x = -y$. Substituting into the constraint, we find that the first has solution $x = \pm 1$ and the second has solution $x = \pm\sqrt{3}$. The resulting critical points are $\pm(1, 1)$ and $\pm(\sqrt{3}, -\sqrt{3})$. The first two minimize the distance ($\sqrt{2}$) and the second two maximize it ($\sqrt{6}$).

18.7 Maximize $f(x, y, z) = yz + xz$ subject to $y^2 + z^2 = 1$ and $xz = 3$.

Answer: #1 (shortcut): We can use the constraint on xz to simplify the objective to $3 + yz$. Since the 3 is irrelevant, we are just maximizing yz subject to the constraint $y^2 + z^2 = 1$.

Then $dh = (2y, 2z) \neq (0, 0)$ since $y^2 + z^2 = 1$, showing that NDCQ is satisfied.

The Lagrangian is $\mathcal{L} = yz + \lambda(y^2 + z^2 - 1)$ and the first order conditions are

$$0 = z + 2y\lambda$$
$$0 = y + 2z\lambda.$$

Eliminating λ, we find $y^2 = z^2 = 1/2$. Then $y = \pm\sqrt{1/2}$, $z = \pm\sqrt{1/2}$. The objective is maximized when both have the same sign, as do x and z, so the maxima occur at $(x, y, z) = \pm(3\sqrt{2}, \sqrt{1/2}, \sqrt{1/2})$ when $yx = 1/2$ and the maximium value is $3 + 1/2 = 3.5$.

#2 (long version): The derivative of the constraints is

$$\begin{bmatrix} 0 & 2y & 2z \\ z & 0 & x \end{bmatrix}.$$

The constraint $xz = 3$ implies $z \neq 0$, and the matrix has rank 2. Constraint qualification holds.

The Lagrangian is $\mathcal{L} = yz + xz - \mu(y^2 + z^2 - 1) - \nu(xz - 3)$. The first-order conditions are

$$\begin{aligned} 0 &= z - \nu z, \\ 0 &= z - 2\mu y, \\ 0 &= x + y - 2\mu z - \nu x. \end{aligned}$$

Since $z \neq 0$, we find $\nu = 1$ from the first equation. The third equation then becomes $y = 2\mu z$. Combining with the second equation, we find $y = 4\mu^2 y$. Now if $y = 0$, $z = 0$, which is impossible. Thus $\mu = 1/2$ or $\mu = -1/2$.

In the first case, $z = y = \pm 1/\sqrt{2}$ and in the second case $z = -y = \pm 1/\sqrt{2}$. The four solutions are $(3\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$ and $(-3\sqrt{2}, -1/\sqrt{2}, -1/\sqrt{2})$, which are both maxima at 3.5, and $(-3\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2})$ and $(3\sqrt{2}, -1/\sqrt{2}, 1/\sqrt{2})$, which both minima at 2.5.

18.13 Show that the budget inequality constraint is binding in Example 18.8 even in the presence of the non-negativity constraints $x_1 \geq 0$, $x_2 \geq 0$. In the process, check the NDCQ for this more general problem.

Answer: Example 18.8 is based on Example 18.1, which is

$$\begin{aligned} &\max U(\mathbf{x}) \\ &\text{s.t } \mathbf{p} \cdot \mathbf{x} \leq I \\ &\mathbf{x} \geq 0 \end{aligned}$$

Example 18.8 considers the case of two goods with $\mathbf{p} \gg 0$. We also assume $DU(\mathbf{x}) \geq 0$ Unlike Example 18.8, we include the non-negativity constraints.

We consider NDCQ first. The matrix formed from the derivatives of the constraint functions is:

$$\begin{pmatrix} p_1 & p_2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

When $I > 0$, at most two of the three constraints can bind simultaneously. If all three were binding, $x_1 = x_2 = 0$, implying that $\mathbf{p} \cdot \mathbf{x} = 0 < I$, showing that the third constraint cannot bind.

Since $p_i > 0$, the rank of any of the 3 matrices formed by deleting one row is 2, as required. The rank of any of the 3 matrices formed by deleting two rows is 1, as required. Thus NDCQ is satisfied.

The Lagrangian is

$$\mathcal{L} = U(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - I) + \mu_1 x_1 + \mu_2 x_2.$$

The first-order conditions are $\partial U/\partial x_1 + \mu_1 = \lambda p_1$, $\partial U/\partial x_2 + \mu_2 = \lambda p_2$. If the budget constraint does not bind, $\lambda = 0$ by complementary slackness. The first-order conditions reduce to

$$\frac{\partial U}{\partial x_1} = -\mu_1 \leq 0$$
$$\frac{\partial U}{\partial x_2} = -\mu_2 \leq 0.$$

It is impossible to satisfy these equations because Example 18.8 assumes that $\partial U/\partial x_i > 0$ for at least one i, which implies $\mu_i < 0$, violating non-negativity. This contradiction shows that the budget constraint must bind.

19.2 Find the maximum of $x + y + z^2$ subject to the constraints $x^2 + y^2 + z^2 = 0.8$, $y = 0$:

a) by using Theorem 19.1 and Exercise 18.6,

b) by doing the calculation from scratch.

Answer:

a) Exercise 18.6 required us to find the maximum of $f(x, y, z) = x + y + z^2$ subject to the constraints $x^2 + y^2 + z^2 = 1$ and $y = 0$.

The derivative of the constraints was

$$\begin{bmatrix} 2x & 2y & 2z \\ 0 & 1 & 0 \end{bmatrix}.$$

Constraint qualification is satisfied because at least one of x and z must non-zero, yielding rank 2.

The Lagrangian was

$$\mathcal{L} = x + y + z^2 - \mu(x^2 + y^2 + z^2 - 1) - \nu y.$$

The first-order conditions were

$$0 = 1 - 2\mu x,$$
$$0 = 1 - 2\mu y - \nu,$$
$$0 = 2z - 2\mu z.$$

Since $y = 0$, the second equation yielded $\nu = 1$. The third equation was $2(1 - \mu)z = 0$.

(*) If $z = 0$, we used the constraint $x^2 + y^2 + z^2 = 1$ to find either $x = 1$ and $\mu = 1/2$ or $x = -1$ and $\mu = -1/2$. The latter it is the minimum, so $x = 1$. Thus $(1, 0, 0)$ was a critical point.

If $z \neq 0$, $\mu = 1$, and $x = 1/2$. Then the constraints imply either $z = \sqrt{3}/2$ or $z = -\sqrt{3}/2$.

There were three critical points with $f(1, 0, 0) = 1$, $f(1/2, 0, \sqrt{3}/2 = 5/4$, and $f(1/2, 0, -\sqrt{3}/2) = 5/4$. The latter two were maxima.

The multiplier $\mu = \partial f/\partial a$ was $\mu = 1$. Since the maximum at $a = 1$ was $5/4$, the new maximum should be approximately $1.25 - .2\mu = 1.05$.

b) The calculation follows part (a) until (*). Again there are three critical points, one with $z = 0$: $(\sqrt{.8}, 0, 0)$ and two with $z \neq 0$, when $\mu = 1$ and $x = 1/2$. Then $z = \pm\sqrt{0.55}$ with $f(1/2, 0, \pm\sqrt{.55}) = 1.05$, so our estimate in part (a) was exactly correct.

19.3 If x thousand dollars is spent on labor and y thousand dollars is spent on equipment, a certain factory produces $Q(x, y) = 50x^{1/2}y^2$ units of output.

a) How should \$80,000 be allocated between labor and equipment to yield the largest possible output?

b) Use Theorem 19.1 to estimate the change in maximum output if this allocation decreased by \$1000.

c) Compute the exact change in b).

Answer:

a) We to maximize $Q = 50x^{1/2}y^2$ subject to the constraint that $x + y = 80$. The Lagrangian is $\mathcal{L} = 50x^{1/2}y^2 - \mu(x + y - 80)$. The resulting first order conditions are

$$\mu = 25x^{-1/2}y^2$$
$$\mu = 100x^{1/2}y.$$

Eliminating μ, we find that $y = 4x$. The solution is $x = 16$, $y = 64$, $\mu = 25600$. The value of output is \$819,200.

b) By Theorem 19.1, the estimated change in the value of output is $\mu \times -1 = -25600$, reducing it to $\$793,600$.

c) We must still spend in a 4-1 ratio, so $x = 15.8$, $y = 63.2$. Substituting in the production function, we find output is now worth $\$793,839.50$. The actual change is $-\$25,360.50$, slightly smaller than the approximation of $-\$25,600$.