Mathematical Economics Exam #1, September 22, 2022

1. Consider the matrix

$$\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

   a) Find all real numbers λ where $\mathbf{A} - \lambda\mathbf{I}$ is singular.

   b) For each λ found in (a), find a non-zero vector $\mathbf{b}$ with $(\mathbf{A} - \lambda\mathbf{I})\mathbf{b} = \mathbf{0}$.

   c) Do the vectors found in part (b) form a basis for $\mathbb{R}^2$?

   Answer:

   a) Set $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and solve for λ to find the λ where $\mathbf{A}$ is singular. Expanding the determinant yields $\lambda^2 - 6\lambda + 8 = 0$, which has solutions $\lambda = 2$ and $\lambda = 4$.

   b) For $\lambda = 2$, we need $\mathbf{b}$ obeying

$$(\mathbf{A} - 2\mathbf{I})\mathbf{b} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\mathbf{b} = \mathbf{0}.$$

   One such vector is $\mathbf{b}_1 = (1, 1)$. For $\lambda = 4$, we need $\mathbf{b}$ obeying

$$(\mathbf{A} - 4\mathbf{I})\mathbf{b} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\mathbf{b} = \mathbf{0}.$$

   One such vector is $\mathbf{b}_2 = (1, -1)$.

   c) We use the determinant test. Since $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0$, the vectors $\{\mathbf{b}_1, \mathbf{b}_2\}$ form a basis for $\mathbb{R}^2$.

2. Consider the following norms on $\mathbb{R}^3$. The ℓ^3 norm $\|\mathbf{x}\|_3 = (|x_1|^3 + |x_2|^3 + |x_3|^3)^{1/3}$, and the sup-norm $\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, |x_3|\}$. (corrected)

   Show that these norms are equivalent on $\mathbb{R}^3$ by finding positive numbers A and B with $A\|\mathbf{x}\|_3 \le \|\mathbf{x}\|_\infty \le B\|\mathbf{x}\|_3$.

   Answer: For each $i = 1, 2, 3$, $|x_i| \le \|\mathbf{x}\|_3$. Then $\|\mathbf{x}\|_\infty = \max_i |x_i| \le \|\mathbf{x}\|_3$. It follows that $B = 1$ works.

   For each $i = 1, 2, 3$, $x_i^3 \le \|\mathbf{x}\|_\infty^3$, so $\|\mathbf{x}\|_3 \le (3\|\mathbf{x}\|_\infty^3)^{1/3} = 3^{1/3}\|\mathbf{x}\|_\infty$. This means that $A = 3^{-1/3}$ will do.

   That gives us $\frac{1}{\sqrt[3]{3}}\|\mathbf{x}\|_3 \le \|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_3$.

3. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 2 & 3 & 4 \\ 1 & 8 & 4 & 9 & 12 \end{pmatrix}$$

   a) Find the reduced row-echelon form of $\mathbf{A}$

   b) Recall $\ker \mathbf{A} = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{0}\}$. What is $\dim \ker \mathbf{A}$?

   c) Find a basis for $\ker \mathbf{A}$.

   Answer:

   a)

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 2 & 3 & 4 \\ 1 & 8 & 4 & 9 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 2 & 3 & 4 \\ 0 & 3 & 2 & 6 & 8 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 5 & 2 & 3 & 4 \\ 0 & 1 & 2/3 & 2 & 8/3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -4/3 & -7 & -28/3 \\ 0 & 1 & 2/3 & 2 & 8/3 \end{pmatrix} =$$

   b) There are 3 free variables, x_3, x_4, and x_5, so $\dim \ker \mathbf{A} = 3$.

   c) We can find a basis systematically by taking $(x_3, x_4, x_5) = (1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. The result is $\mathbf{b}_1 = (4/3, -2/3, 1, 0, 0)$, $\mathbf{b}_2 = (7, -2, 0, 1, 0)$, $\mathbf{b}_3 = (28/3, -8/3, 0, 0, 1)$.

4. Find all vectors in $\mathbb{R}^4$ that are perpendicular to

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 3 \\ 1/3 \\ 0 \\ 1 \end{pmatrix}.$$

   Answer: Such vectors $\mathbf{z}$ must obey $\mathbf{x}_1 \cdot \mathbf{z} = 0$ and $\mathbf{x}_2 \cdot \mathbf{z} = 0$. We can write this as a system of linear equations:

$$x_1 + x_2 + x_3 = 0$$
$$3x_1 + \frac{1}{3}x_2 + x_4 = 0$$

   To solve this homogeneous system, we row-reduce the coefficient matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 1/3 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -8/3 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 9/8 & -3/8 \end{pmatrix}.$$

There are two free variables (x_3, x_4) and two basic variables. The solutions to this homogeneous system are the vectors perpendicular to $\mathbf{x}_1$ and $\mathbf{x}_2$. The kernel has dimension 2. We can describe the kernel in terms of basis vectors which we find by setting $(x_3, x_4) = (1, 0)$ and $(x_3, x_4) = (0, 1)$. The result is $\mathbf{b}_1 = (1/8, -9/8, 1, 0)$ and $\mathbf{b}_2 = (-3/8, 3/8, 0, 1)$. A vector is perpendicular to both $\mathbf{x}_1$ and $\mathbf{x}_2$ if and only if it is a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$.