Mathematical Economics Exam #2, October 27, 2022

1. Find the infinite Taylor series at $a = 0$ for e^x. Show that it converges for every x.

   Answer: Each derivative of e^x is e^x. Now $e^a = e^0 = 1$. It follows that the required Taylor series is
$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$
   We now apply the ratio test for convergence. Let $b_n = x^n/n!$ be the nth term in the series. Then
$$\limsup_{n\to+\infty} \left|\frac{b_{n+1}}{b_n}\right| = \limsup_{n\to+\infty} \left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \limsup_{n\to+\infty} \left|\frac{x}{n+1}\right| = 0$$
   for every $x \in \mathbb{R}$. Since $0 < 1$, the series converges at every point x. In fact, it converges uniformly.

2. Let $f(x_1, x_2, x_3) = \mathbf{p}\cdot\mathbf{x} + x_1^2 + x_2^2 + x_1x_2 + x_1x_2x_3$ where $\mathbf{p} \in \mathbb{R}^3$. Compute both the Fréchet derivative Df and the Hessian D^2f. Is the Hessian symmetric?

   Answer: Since $f\colon \mathbb{R}^3 \to \mathbb{R}$, the Fréchet derivatives also maps $\mathbb{R}^3 \to \mathbb{R}$. In other words, it should be written as s row vector (covector). The derivative Df is the covector
$$Df = (p_1 + 2x_1 + x_2 + x_2x_3, p_2 + 2x_2 + x_1 + x_1x_3, p_3 + x_1x_2).$$
   To obtain the Hessian, we take the second derivatives downward from Df, yielding
$$D^2f = \begin{pmatrix} 2 & 1+x_3 & x_2 \\ 1+x_3 & 2 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}.$$
   as the Hessian.

   Yes, the Hessian is symmetric.

3. Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ e^{-1/x^2} & \text{for } x > 0 \end{cases}$$
   a) Show that f is continuous at $x = 0$.

   b) Show that f' is continuous at $x = 0$.

   c) Show that f'' is continuous at $x = 0$.

Answer:

a) For $x < 0$, $f(x) = 0$. Suppose $|x| < \delta$. Then $-\delta < x < \delta$ and $0 < x^2 < \delta^2$. It follows that $1/x^2 > 1/\delta^2$ and $e^{-1/x^2} < e^{-1/\delta^2}$. So whenever $|x| < \delta$, we can conclude $|f(x) - 0| = |f(x)| < e^{-1/\delta^2}$. Then for $0 < \varepsilon < 1$, setting $\delta = 1/\sqrt{|\ln \varepsilon|}$ implies$|f(x) - f(0)| = |f(x)| < \varepsilon$. It follows that f is continuous at $x = 0$.

b) For $x < 0$, $f'(x) = 0$. For $x > 0$, $f'(x) = (2/x^3)e^{-1/x^2}$. The limit from both sides is zero because e^{-1/x^2} goes to zero faster than any polynomial in $1/x$ goes to ∞. That means f' is continuous at 0.

The following method can be used to show the limit of f' from the right is zero. First replace $1/x^2$ by u. Here $u \to \infty$. Then we rewrite f' in terms of u, rewrite again so that the numerator and denominator both converge to $+\infty$, and use l'Hôpital's rule twice to see that the limit is zero. In detail,

$$\lim_{x\to 0} \frac{2e^{-1/x^2}}{x^{-3}} = \lim_{u\to\infty} 2u^{3/2}e^{-u} = \lim_{u\to\infty} \frac{2u^{3/2}}{e^u} = \lim_{u\to\infty} \frac{3u^{1/2}}{e^u} = \lim_{u\to\infty} \frac{3u^{-1/2}}{2e^u} = 0.$$

c) As in (b), $f''(x) = 0$ for $x < 0$. For $x > 0$, $f''(x) = \left[(4/x^6) - (6/x^4)\right]e^{-1/x^2}$. A calculation as in (b) shows the limit as $x \to 0$ is zero.

4. Let C_0 be the closed interval $[0, 1]$. Define C_1 by removing the open middle third, $(\frac{1}{3}, \frac{2}{3})$, so $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. To obtain C_2, we do the same thing to each interval in C_1, so that

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

Let C_n be a finite union of closed subintervals of $[0, 1]$, written $C_n = \cup_{k=1}^{k_1} F_{n,k}$. Thus $C_0 = F_{0,1}$, $C_1 = F_{1,1} \cup F_{1,2}$, and $C_2 = F_{2,1} \cup F_{2,2} \cup F_{2,3} \cup F_{2,4}$.

Given C_n, inductively define C_{n+1} to be the set formed by removing the open middle third from each of subintervals F_{nk}, as we did with C_1 and C_2.

a) Determine how many subintervals are in C_n and show that each C_n is the union of finitely many closed disjoint subintervals.

b) How long are the subintervals in C_n and what is their total length?

c) Define $\mathfrak{C} = \cap_{n=0}^{\infty} C_n$. Show that $\mathfrak{C}$ has an empty interior.

Answer:

a) Note that both C_0 and C_1 are the union of finitely many closed subintervals of $[0, 1]$. There is one subinterval for $C_0 = [0, 1]$ and 2 for C_1. Suppose $C_n = \cup_{k=1}^{k_n} F_{n,k}$ where the $F_{n,k}$ are disjoint for $k = 1, \ldots, k_n$. Removing the open middle third breaks each subinterval into two subsubintervals. Since we removed the open middle third, both of the subsubintervals are closed. It follows that $k_{n+1} = 2k_n$ so $C_{n+1} = \cup_{k=1}^{k_{n+1}} F_{n+1,k}$ is the union of $2k_n$ closed intervals $F_{n+1,k}$. Since this is true of C_0 with $n_0 = 1$, it is true of all C_n with $n_k = 2^k$ by induction.

b) Each time we remove the middle third from $F_{n,k}$, we create two subintervals with one third the length of $F_{n,k}$. Now $F_{0,0} = [0,1] = C_0$ has length one, so the intervals $F_{n,k}$ all have length $(1/3)^n$. There are 2^n subintervals, so the total length of the subintervals of C_n is $(2/3)^n$. This shrinks to zero are $n \to \infty$.

c) The set $\mathfrak{C}$ is the well-known Cantor set. Recall that at step n, there are 2^n subintervals of length $(1/3)^n$ that comprise $C_n \subset \mathfrak{C}$. Suppose there is $x \in \mathfrak{C}$ and $\varepsilon > 0$ with $B_\varepsilon(x) \subset \mathfrak{C}$. Then for each n there is some k_n with $B_\varepsilon(x) \subset F_{n,k_n}$. But then, $2\varepsilon < (1/3)^n$ for all n, which is impossible. The ε ball cannot fit in $F_{n,k}$ for large n. It follows that the interior of $\mathfrak{C}$ is empty.