Mathematical Economics #2 Midterm Extra Credit

November 4, 2022

1. Find the infinite Taylor series at $a = \pi$ for $\sin x$. Show that it converges for every x.

   Answer: The general Taylor series formula at a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The derivatives of $f(x) = \sin x$ are $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, etc. When evaluated at $x = \pi$, we have $f(\pi) = 0$, $f'(\pi) = -1$, $f''(\pi) = 0$, $f'''(\pi) = +1$, after which the pattern starts over again.

It follows that the required Taylor series is

$$-(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \cdots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\pi)^{2n+1}}{(2n+1)!}$$

We now apply the ratio test for convergence. Let $b_n = (-1)^{n+1}(x-\pi)^{2n+1}/(2n+1)!$ be the nth term in the series. Then

$$\begin{aligned}
\limsup_{n\to+\infty} \left|\frac{b_{n+1}}{b_n}\right| &= \limsup_{n\to+\infty} \left|\frac{(x-\pi)^{2n+3}}{(2n+3)!}\frac{(2n+1)!}{(x-\pi)^{2n+1}}\right| \\
&= \limsup_{n\to+\infty} \left|\frac{(x-\pi)^2}{(2n+3)(2n+2)}\right| \\
&= 0
\end{aligned}$$

for every $x \in \mathbb{R}$. Since $0 < 1$, the series converges at every point x. In fact, it converges uniformly.

2. Let $f(x_1, x_2, x_3) = \mathbf{p}\cdot\mathbf{x} + e^{x_1^2} + x_1^2 x_2 + x_1 x_2 x_3^3$ where $\mathbf{p} \in \mathbb{R}^3$. Compute both the Fréchet derivative Df and the Hessian $D^2 f$. Is the Hessian symmetric?

   Answer: Since $f\colon \mathbb{R}^3 \to \mathbb{R}$, the Fréchet derivatives also maps $\mathbb{R}^3 \to \mathbb{R}$. In other words, it should be written as a row vector (covector). The derivative Df is the covector

$$Df = (p_1 + 2x_1 e^{x_1^2} + 2x_1 x_2 + x_2 x_3^3,\, p_2 + x_1^2 + x_1 x_3^3,\, p_3 + 3x_1 x_2 x_3^2).$$

To obtain the Hessian, we take the second derivatives downward from Df, yielding

$$D^2f = \begin{pmatrix} (2+4x_1^2)e^{x_1^2}+2x_2 & 2x_1+x_3^3 & 3x_2x_3^2 \\ 2x_1+x_3^3 & 0 & 3x_1x_3^2 \\ 3x_2x_3^2 & 3x_1x_3^2 & 6x_1x_2x_3 \end{pmatrix}.$$

as the Hessian.

Yes, the Hessian is symmetric.

3. Define the function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = e^{-x^2}.$$

a) Show that $\lim_{x\to+\infty} f(x) = 0$. Here L is the limit of $f(x)$ at $+\infty$ means that for every $\varepsilon > 0$ there is a $K > 0$ with $|f(x) - L| < \varepsilon$ whenever $x > K$. You should use the formal definition in your answer to this part (not needed in the others).

b) Show that $\lim_{x\to+\infty} x^\alpha f(x) = 0$ for any real number α. For $\alpha > 0$, the problem here is that $x^\alpha \to +\infty$ while $f(x) \to 0$, so naively taking the product of the limits you would obtain $0\cdot\infty$, which is undefined.

c) Let $p(x) = a_n + a_{n-1}x + \cdots + a_0x^n$ be a polynomial. Show that $\lim_{x\to+\infty} p(x)f(x) = 0$.

Answer:

a) Let $1 > \varepsilon > 0$. We must find a K so that $f(x) < \varepsilon$ for all $x > K$. Now suppose $x > K = \sqrt{-\ln\varepsilon}$. Then $x^2 > -\ln\varepsilon$, so $-x^2 < \ln\varepsilon$. Then $f(x) = e^{-x^2} < \varepsilon$ for $x > K$.

b) We first simplify by substituting $x^2 = u$, so we want to calculate $\lim u^{\alpha/2}e^{-u}$. We still have the $0\cdot\infty$ problem, but this will make our calculations easier.

We rewrite

$$u^{\alpha/2}e^{-u} = \frac{x^{\alpha/2}}{e^u}.$$

Now the numerator and denominator both converge to $+\infty$, which allows us to apply l'Hôpital's rule which states that the limit is the limit of the ratio of the limit of the derivatives of numerator and denominator:

$$\frac{(\alpha/2)u^{\alpha/2-1}}{e^u}.$$

If $\alpha/2 > 1$, we still have the problem, but if $\alpha < 2$, the numerator converges to zero, showing $x^\alpha e^{-x^2} \to 0$.

If we apply l'Hôpital's rule n times, we obtain

$$\lim \frac{(\alpha/2)(\alpha/2-1)\cdots(\alpha/2-n+1)u^{\alpha/2-n}}{e^u},$$

which is zero for $\alpha < 2n$. By applying l'Hôpital's rule n times with $n > \alpha/2$, we show the limit is $0/\infty = 0$.

$c)$ Since the limit of the sum is the sum of the limits, we can apply part (b) separately to each term $a_{n-k}x^k e^{-x^2}$ for $k = 0, \ldots, n$. Each term converges to zero, and so does the sum. This shows that e^{-x^2} converges to 0 as $x \to +\infty$ faster than $|p(x)|$ grows at $x \to +\infty$ for any polynomial $p(x)$.