6. Intro to Linear Equations and Systems

8/25/20

6.1 Sample Linear Equations

We start with some sample linear equations.

$$\begin{aligned} x &= 2 && \text{a point in } \mathbb{R} \\ 2x &= 3y + 7 && \text{a line in } \mathbb{R}^2 \\ 10 &= x + 3y + 4z && \text{a plane in } \mathbb{R}^3 \\ 0 &= 0 && \text{anything is a solution!} \end{aligned}$$

Here $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^n$ is the set of all n-tuples of real numbers, $(x_1, \ldots, x_n)$, with each $x_i \in \mathbb{R}$.

6.2 Equations that are Not Linear

Here are some equations that are not linear equations.

$$
\begin{aligned}
y &= e^x \\
x_3 &= \cos x_1 + 3\sin^2 x_2 \\
x_2 &= 2x_2^2 + 3x_2 + 12 \\
z &= 7xy + 5y + 2 \\
z &= 3 + \sqrt{x + xy + y^2} \\
y &= \sqrt{x^2}
\end{aligned}
$$

For the last equation, keep in mind that it is the positive square root. That equation can also be written $y = |x|$.

6.3 Linear Equations

The key thing about linear equations is that they are linear in each variable. You can't take powers or other functions of variables, nor multiply them together. All you can multiply a variable by is a constant. More formally, a *linear equation* in n variables $x_1, x_2, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

for some real numbers a_i, $i = 1, \dots, n$ and b. We can use summation notation to write this equation in the more compact form

$$\sum_{i=1}^{n} a_ix_i = b$$

You'll notice that there is only one variable in each term, and that it always is merely itself, never a function.

6.4 What are the Solutions to Linear Equations?

Suppose we have a linear equation in n variables:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{6.4.1}$$

Consider the solution set,

$$\{(x_1, \ldots, x_n) \text{ that solve equation (6.4.1)}\}$$

What does the solution set look like?

If we have 2 variables, a linear equation is the equation of a straight line. With 3 variables, equation (6.4.1) describes a plane. In general, with n variables, the solutions to equation (6.4.1) form a hyperplane, an $(n-1)$-dimensional vector subspace of $\mathbb{R}^n$. For all $n \geq 2$, if you have two distinct points in the solution set, the line they generate will also be in the solution set.

Don't ask me to draw them!

6.5 Linear Systems

A *linear system* in variables $x_1, \ldots, x_n$ is a collection of linear equations. Let a_{ij}, b_i be real numbers with $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We can write the system as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{6.5.2}$$

One consequence of linearity is that if $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ both solve the linear system (6.5.2), their difference solves the linear system with the same coefficients but with each $b_i = 0$. This is referred to as the associated *homogeneous system*.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= 0 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

If $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ solve the homogeneous system, so does any linear combination of them.

By the linearity of the system, if we have one solution $(x^*_1, \ldots, x^*_n)$ to equation (6.5.2), any other solution can be written as the sum of it and a solution to the associated homogeneous system. This property is shared with other types of linear systems, such as systems of linear differential equations.

6.6 What are the Solutions to Linear Systems?

As with linear equations, we ask what the solution set of a linear system is. What does $\{(x_1, \ldots, x_n) : (x_1, \ldots, x_n)$ solves equation (6.5.2)$\}$ look like?

To help build intuition, consider the case $n = 2$. If they are non-trivial, the linear equations that make up the system (6.5.2) each describe lines in $\mathbb{R}^2$. If it satisfies each equation, a point (x_1, x_2) must be on every line, meaning that it is in the intersection of a bunch of lines. The intersection might be empty, it might be a single point, or it might contain two points.

If it contains two points, these points determine a straight line, and that line must be the straight line described by each of the linear equations in system (6.5.2). That whole line must be the intersection.

There is one more case to consider. What if the equations are trivial? In that case, they impose no restrictions, as in the system

$$\begin{aligned} 0 &= 0 \\ 42 &= 42 \\ 137 &= 137. \end{aligned}$$

If this is a system in n variables, anything in $\mathbb{R}^n$ solves it.[1]

It follows that in $\mathbb{R}^2$ the intersection that is the solution set is either empty, a single point, a straight line, or the entirety of $\mathbb{R}^2$.

In $\mathbb{R}^3$, the solution set is the intersection of planes. It can be empty, a point, a straight line, a plane, or the whole space.

The possibilities are similar in higher dimensions. They are always the intersections of hyperplanes, or else the whole space.

[1] The numbers come from Brahmagupta (earliest known use of the number zero), Douglas Adams (the ultimate answer), and Arnold Sommerfield (fine structure constant).

6.7 Example: Taxes and Charitable Deductions

Let's examine a simple linear system.

Suppose a company has before-tax profits of \$100,000. It will contribute 10% of its after-tax profits to the Red Cross. It pays a state tax equal to 5% of its post-contribution profit and a federal tax of 40% of its profit post contribution and state tax. How much does the company pay in federal taxes.

We can set this up as a linear system. Let C be the charitable contribution, S be the state tax, and F the federal tax. After-tax profits are $100,000 - (S + F)$, so $C = 0.1\big(100,000 - (S + F)\big)$. We rewrite this as

$$C + 0.1S + 0.1F = 10,000$$

The state tax is 5% of the profit net of the donation. Then $S = 0.05(100,000 - C)$ or

$$0.05C + S = 5,000.$$

Federal taxes are 40% of the remaining profit, $F = 0.40\big(100,000 - (C + S)\big)$. In other words $0.4C + 0.4S + F = 40,000$.

We put these three equations together to form our linear system:

$$\begin{aligned} C + 0.1S + 0.1F &= 10,000 \\ 0.05C + S &= 5,000 \\ 0.4C + 0.4S + F &= 40,000. \end{aligned}$$

By solving this system of three equations in three unknowns, we can calculate the charitable contribution (C) as well as the state (S) and federal (F) taxes.

6.8 Example: Input-Output Model I

Input-output models provide another example of a linear system. The basic model includes a primary good, which we will refer to as labor and denote by index 0, and n produced goods. The produced goods can either be used as inputs to production or consumed and are labeled $1, \dots, n$.

Production involves both *fixed proportions* and *constant returns to scale*. Fixed proportions means that the inputs are always combined in the same ratio, as hydrogen and oxygen are always combined in a 2:1 ratio to make water, H_2O (or a 1:1 ratio to make hydrogen peroxide H_2O_2). Constant returns means that scaling all inputs by a factor $\alpha > 0$ scales output by that same factor α.

For example, suppose that the production of one 8 ounce glass of chocolate milk requires one 8 ounce glass of milk, two teaspoons of chocolate mix, and two minutes of labor, to make it well-stirred.

Then two glasses of chocolate milk can be produced using two glasses of ordinary milk, four teaspoons of chocolate mix, and four minutes of labor (doubling all inputs doubles the output).

Let a_{ij} be the amount of input i required to produce one unit of good j. These are referred to as the *input-output coefficients*. In our chocolate milk example, we let good 1 be ordinary milk, good 2 be chocolate mix, and good 3 be chocolate milk. The input coefficients are $a_{03} = 1/30$, $a_{13} = 1$, $a_{23} = 2$, and $a_{33} = 0$, where labor is measured in hours, both types of milk in 8 ounce glasses, and chocolate mix in teaspoons.

6.9 Example: Input-Output Model II

Suppose we have input-output coefficients a_{ij} and want to produce x_j units of good j for $j = 1, \ldots, n$. This requires input of

$$\sum_{j=1}^{n} a_{ij} x_j$$

units of good i for $i = 0, \ldots, n$. We can think of this expression as the demand for good i by producers. Note that labor is included ($i = 0$).

Consumers consume goods $i = 1, \ldots, n$. The quantity demanded of good i is c_i for $i = 1, \ldots, n$. The consumers do not demand labor. They supply it.

Finally, suppose the economy has L units of labor which is supplied by consumers. If we want to produce enough so that consumers can consume $c_1, \ldots, c_n$, we must find non-negative values of the x_i so that

$$\begin{aligned}
\text{Supply} &= \text{Producer Demand} + \text{Consumer Demand} \\
L &= a_{01}x_1 + a_{02}x_2 + \cdots + a_{0n}x_n \\
x_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + c_1 \\
x_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + c_2 \\
\vdots & \qquad\qquad\quad \vdots \\
x_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + c_n.
\end{aligned}$$

This linear system is the *Leontief system*. Basically, it says that supply is equal to the sum of producer and consumer demand in every market. This means it is a primitive general equilibrium model, where all markets must simultaneously clear. (Primitive, because it lacks prices.)

6.10 Example: Markovian Employment Model I

This model has two possible states, employed and unemployed. We don't distinguish how long someone has been either employed or unemployed.

If someone is unemployed, they have a probability p, $0 \leq p \leq 1$ of finding a job this week. There is a $(1 - p)$ chance they don't find a job and remain unemployed.

If someone is employed, they have a probability q, $0 \leq q \leq 1$ of losing that job this week and becoming unemployed. There is a probability $(1 - q)$ of remaining employed.

Models where the probability of moving from one state to another depends only on the current state are called *Markovian*. The probabilities of moving between the various states are called *transition probabilities*.

Now suppose x people are currently employed and y are unemployed. Next week, $qx + (1 - p)y$ will be unemployed next week, and $(1 - q)x + py$ will be employed next week. These quantities always sum to $x + y$, the total labor force.

6.11 Example: Markovian Employment Model II

We now consider a sequence of weeks, indexed by time t. Then our system becomes

$$\begin{aligned} x_{t+1} &= (1-q)x_t + py_t \\ y_{t+1} &= qx_t + (1-p)y_t. \end{aligned}$$

This is a case where we have a linear system of *difference equations*. These equations can be rewritten so the differences $x_{t+1} - x_t$ and $y_{t+1} - y_t$ are functions of (x_t, y_t).

Once again, $x_t + y_t$ remains constant. We say that (x, y) is a *stationary distribution* if $(x_t, y_t) = (x, y)$ implies that $(x_{t+1}, y_{t+1}) = (x, y)$. In other words, once we reach a stationary distribution, the system no longer evolves over time. It remains at (x, y).

Let's find the stationary distribution under the assumption that $x + y = N$. We will also require that $p > 0$. The stationary distribution must solve the following equations.

$$x = (1-q)x + py \tag{6.11.3}$$
$$y = qx + (1-p)y \tag{6.11.4}$$
$$x + y = N. \tag{6.11.5}$$

We will solve for (x, y). Before proceeding, note that equations (6.11.3) and (6.11.4) are different forms of the same equation. Both can be simplified to $qx = py$. This means that it is enough to satisfy (6.11.3) and (6.11.5) as (6.11.4) automatically follows.

Now we rewrite equation (6.11.3) so that $qx = py$. Since $p > 0$,

$$y = \frac{q}{p}x.$$

We plug this into (6.11.5), obtaining $x + qx/p = N$. Then

$$x\left(\frac{p+q}{p}\right) = N$$

so

$$x = \frac{p}{p+q}N \quad \text{and} \quad y = \frac{q}{p+q}N.$$

We have found the stationary distribution by solving the system (6.11.3)–(6.11.5).

6.12 Example: IS-LM Model

A simple IS-LM model without trade provides another example of a linear system. Let Y denote GDP, C consumption, I investment by firms, and G government spending. With all of GDP accounted for,

$$Y = C + I + G.$$

We use a simplified Keynesian consumption function, where $C = bY$ with $0 < b < 1$. Here b is the *marginal propensity to consume* and $s = (1 - b)$ is the *marginal propensity to save*. Firms' investment depends on the interest rate r. We expect it to be decreasing in r because higher interest rates decrease the present value of future income. We write

$$I = I^0 - ar.$$

The parameter a is called the *marginal efficiency of capital*. We can combine these equations to obtain the IS (investment-saving) curve. Here

$$Y = bY + (I^0 - ar) + G,$$

so

$$sY + ar = I^0 + G \tag{IS}$$

The other half of the model is the LM (liquidity-money) equation, which characterizes money market clearing. Money demand comes in two parts. One is from use in transactions, and is proportional to GDP. The other part is speculative demand. Investors must decide whether to hold bonds or cash. The interest rate negatively affects the speculative demand. Thus money demand is $M_d = mY + (M^0 - hr)$ where m, h, and M^0 are all positive.

Money supply is fixed at M_s in this model. We set supply equal to demand

$$M_s = mY + M^0 - hr$$

and rearrange to obtain the LM equation

$$mY - hr = M_s - M^0 \tag{LM}$$

The IS-LM model is formed by combining the IS and LM equations.

$$\begin{aligned} sY + ar &= I^0 + G \\ mY - hr &= M_s - M^0. \end{aligned}$$

We have two equations in the two unknowns Y and r. Here $I^0 + G$ and $M_s - M^0$ are the constant terms and s, a, m, and $-h$ the coefficients.

7. Solving Linear Systems

There are three key questions concerning the solution of linear systems:

1. Existence: Is there a solution?
2. Uniqueness: Is there one solution or many?
3. How can we find all the solutions?

7.1 Methods of Solution of Linear Systems

Let's consider a linear system with m equations in n variables:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1, \ldots, m.$$

Three methods have proven useful to solve linear systems.

1. Substitution. As used in the Markovian employment model.
2. Elimination of Variables. The Gauss and Gauss-Jordan methods
3. Matrix Algebra: Cramer's Rule.

We already saw substitution at work in the Markovian employment model. It can be a quick way to solve linear systems when they have the right kind of structure, but in general, it may be difficult to make progress using substitution.

7.2 Rewriting the Equation System

Gauss and Gauss-Jordan elimination is based on the fact that the equations of a linear system may be manipulated in certain ways without affecting the solutions of the system.

Consider the linear system

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3\\
\vdots \qquad\qquad\qquad & \quad \vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

We refer to the i^{th} equation as E_i and the whole system (E_1)–(E_m) as E. We will be interested in ways of altering the equations that do not change the solutions to the system. The following theorem forms the basis for the Gauss-Jordan method. It tells us that certain changes to the system of equations do not change the solution set. We can reorder equations or use either of two linear operations: adding equations and multiplying by a non-zero number.

Theorem 7.2.1. *Suppose a linear system* $E = (E_1)$–(E_m) *is altered by either*

1. *changing the order of the equations,*
2. *multiplying an equation by a non-zero constant, or*
3. *adding one equation to another.*

Then the set of solutions to the system is not changed.

In other words, none of our three basic equation operations affect the solution set.

7.3 Proof of Theorem 7.2.1

Theorem 7.2.1. *Suppose a linear system $E = (E_1)$–(E_m) is altered by either*

1. *changing the order of the equations,*
2. *multiplying an equation by a non-zero constant, or*
3. *adding one equation to another.*

Then the set of solutions to the system is not changed.

Proof. (1) Consider the set of solutions of equations $E = (E_1) - (E_m)$. If we change the order of equations, we still must satisfy $(E_1) - (E_m)$, so the solutions have not changed.

(2) Suppose we multiply equation i by $\alpha \neq 0$. Then we have a new linear equation

$$\alpha a_{i1}x_1 + \alpha a_{i2}x_2 + \cdots \alpha a_{in}x_n = \alpha b_i \tag{E_i'}$$

Consider the new system E' defined by $E_j' = E_j$ for $j \neq i$ with E_i' as above. This new system has the same solutions as the old system. Clearly, if $(x_1, \ldots, x_n)$ satisfy the old system, they obey each of the new equations (only E_i' is different), and if $(x_1, \ldots, x_n)$ satisfies the new system, the fact that $\alpha \neq 0$ allows us to divide, showing that E_i is also satisfied. Since the other equations are the same, the old system E is also satisfied.

(3) Suppose we replace equation (E_i) with the sum of (E_i) and (E_j). Call this equation (E_i') and define $E_k' = E_k$ for $k \neq i$ to obtain a new system E'. If $(x_1, \ldots, x_n)$ solves E, it must also solve E_i', so E' is satisfied. Conversely, if E' is satisfied by $(x_1, \ldots, x_n)$, so is the equation formed by subtracting $E_j' = E_j$ from E_i'. But that is just E_i, so E is also satisfied. □

7.4 Corollary to Theorem 7.2.1

The result is sometimes stated in a slightly different form, where we add a multiple of one equation to another. But we can get there from Theorem (7.2.1) in a couple of steps. First, we multiply the one equation by $\alpha \neq 0$, in step two we add it to the other, in step three, we divide the first equation by α, leaving us with a system where the only change to replace the second equation by the sum of α times the first equation and the second equation. We restrict our attention to non-zero α because the $\alpha = 0$ case leaves everything unchanged.

That gives us the following corollary.

Corollary 7.4.1. *Suppose a linear system* $E = (E_1)$–(E_m) *is altered by either*

1. *interchanging two equations,*
2. *multiplying an equation by a non-zero constant, or*
3. *replacing one equation by its sum with a non-zero multiple of another.*

Then the set of solutions to the system is not changed.

The three equation operations can be referred to as *elementary equation operations*. When we later put them into a matrix context, they will become the elementary row operations. In (1), changing the order of equations was replaced by swapping any two equations. That basic operation can be used to generate any change in their order.

7.5 Gauss-Jordan Elimination

The Gauss-Jordan procedure starts by arranging the equations so that the first non-zero term in the top equation is as far to the left as possible.

Step 2 is to divide that equation by the first non-zero coefficient, resulting in an equation where the leading non-zero coefficient is one.

Step 3 is referred to as the *pivot* step. Multiples of the top equation are subtracted from each of the other equations (as needed) to zero out the coefficient on the pivot variable in those equations, leaving just one equation (the top one) where the pivot variable is non-zero (in fact, one).

Finally, we keep repeating this procedure on the remaining equations until none are left. In the end, we will have a simplified system of equations telling us the value of various variables, sometimes in terms of other variables.

7.6 Gauss-Jordan in Action

Consider the system

$$\begin{aligned} 2x_1 - 2x_2 &= 10 && (1) \\ 3x_1 + 2x_2 &= 3 && (2) \end{aligned}$$

We start by making sure the top equation has a leading non-zero coefficient as far left as possible. Since x_1 has coefficient 2, this is true, and we do not have to rearrange the equations. Next, we divide equation (1) by 2 obtaining

$$\begin{aligned} x_1 - x_2 &= 5 && (1') \\ 3x_1 + 2x_2 &= 3 && (2) \end{aligned}$$

where equation $(1') = \frac{1}{2} \times (1)$.

Next comes the pivot step, where we subtract 3 times equation (1′) from equation (2).

$$\begin{aligned} x_1 - x_2 &= 5 && (1') \\ 0x_1 + 5x_2 &= -12. && (2') \end{aligned}$$

Notice that $(2') = (2) - 3 \times (1')$.

We now repeat the procedure on the remaining equations. In this case, there is only one such. Since the coefficients on x_1 are zero in the remaining equations due to the pivot step, we look for a non-zero coefficient on x_2, and there it is!

We then divide (2′) by that coefficient, 5, obtaining $(2'') = (2')/5$:

$$\begin{aligned} x_1 - x_2 &= 5 && (1') \\ x_2 &= -12/5. && (2'') \end{aligned}$$

We finish the problem off by pivoting on x_2, $(1'') = (1') + (2'')$:

$$\begin{aligned} x_1 &= 13/5 && (1'') \\ x_2 &= -12/5. && (2'') \end{aligned}$$

This system is now in a form where we can read off the solution, $(x_1, x_2) = (13/5, -12/5)$.

It is often a good idea to plug these back in the original system to make sure you have not made any calculation errors, something that is far too easy to do.

$$\begin{aligned} 2x_1 - 2x_2 &= 26/5 + 24/5 = 10 \\ 3x_1 + 2x_2 &= 39/5 - 24/5 = 3 \end{aligned}$$

Our answers passed the test. We made no mistake in deriving them.

7.7 Equations as Number Arrays

As you have seen, even a simple Gauss-Jordan calculation involves a lot of writing. Things can be simplified by converting the equations to a rectangular array of numbers, a *matrix*.

Suppose we start with the system

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 10 \\ 5x_1 + 2x_2 + 2x_3 &= 5 \end{aligned} \tag{7.7.1}$$

The x_i's kept getting repeated. As long as we write them in the same order, it is enough to keep track of the coefficients. We replace the system by

$$\begin{pmatrix} 2 & 3 & 1 & \vdots & 10 \\ 5 & 2 & 2 & \vdots & 5 \end{pmatrix},$$

where we have used vertical dots to separate the b_i from the coefficients. Column one contains the x_1 coefficients, column two the x_2 coefficients, etc.

We refer to

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 2 & 2 \end{pmatrix}$$

as the *coefficient matrix* and

$$\hat{\mathbf{A}} = \begin{pmatrix} 2 & 3 & 1 & 10 \\ 5 & 2 & 2 & 5 \end{pmatrix}$$

as the *augmented matrix*. Here the term *matrix* means that we have a rectangular array of numbers. If $\mathbf{A}$ is the matrix of coefficients, we denote the corresponding augmented matrix by $\hat{\mathbf{A}}$.

7.8 Coefficient and Augmented Matrices

In the general case, we have a linear system of m equations in n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

The *coefficient matrix* is the $m \times n$ matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

and the *augmented matrix* is the $m \times (n + 1)$ matrix

$$\hat{\boldsymbol{A}} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{pmatrix}.$$

7.9 Simplifying the Calculations

We can use the augmented matrix to simplify the calculations of the Gauss-Jordan method by keeping track of the variables by position (column) rather than continually writing down the variables. When we add or multiply equations we are adding or multiplying the coefficients, including the constant term. The elementary equation operations become the *elementary row operations* in the augmented matrix: swapping any two rows, multiplying a row by a non-zero constant, and adding a non-zero multiple of one row to another.

Let's try this with the augmented matrix $\hat{\mathbf{A}}$ given above.

$$\begin{aligned}
\hat{\mathbf{A}} = \begin{pmatrix} 2 & 3 & 1 & 10 \\ 5 & 2 & 2 & 5 \end{pmatrix} &\xrightarrow{(1)/2} \begin{pmatrix} 1 & 3/2 & 1/2 & 5 \\ 5 & 2 & 2 & 5 \end{pmatrix} \\
&\xrightarrow{(2)-5(1)} \begin{pmatrix} 1 & 3/2 & 1/2 & 5 \\ 0 & -11/2 & -1/2 & -20 \end{pmatrix} \\
&\xrightarrow{-\frac{2}{11}(2)} \begin{pmatrix} 1 & 3/2 & 1/2 & 5 \\ 0 & 1 & 1/11 & 40/11 \end{pmatrix} \\
&\xrightarrow{(1)-(3/2)(2)} \begin{pmatrix} 1 & 0 & 4/11 & -5/11 \\ 0 & 1 & 1/11 & 40/11 \end{pmatrix}.
\end{aligned}$$

At this point, we have run out of rows. Let's write down the equations.

$$\begin{aligned}
x_1 + \frac{4}{11}x_3 &= -\frac{5}{11} \\
x_2 + \frac{1}{11}x_3 &= \frac{40}{11}.
\end{aligned}$$

We have two equations with three unknowns. With this system, every choice of x_3 (or x_1 or x_2) will give us a different solution. Any combination

$$(x_1, x_2, x_3) = \left(-\frac{5 + 4x_3}{11}, \frac{40 - x_3}{11}, x_3\right)$$

is a solution.

By plugging back into the original system (7.7.1), we can verify that they all work. This system not only has a solution, it has infinitely many solutions!

The solution set is a line in $\mathbb{R}^3$.

7.10 A System without a Solution

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The linear system

$$\begin{aligned} x_1 + 3x_2 &= 6 \\ 2x_1 + 6x_2 &= 10 \end{aligned}$$

has no solution.

To see this, we attempt to find a solution by forming the augmented matrix and row-reducing.

$$\begin{pmatrix} 1 & 3 & 6 \\ 2 & 6 & 10 \end{pmatrix} \xrightarrow{(2)-2(1)} \begin{pmatrix} 1 & 3 & 6 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}\times(2)} \begin{pmatrix} 1 & 3 & 6 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{(1)-6(2)} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This row-echelon form is a problem. To see it, write out the corresponding system of equations

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ 0 &= 1. \end{aligned}$$

The second equation cannot possibly be true. That means the system has no solutions.

7.11 Row-Echelon Form

A row of a matrix has k *leading zeros* if the first k elements of the row are zero and the $(k+1)^{st}$ element is not zero. If a row starts with a non-zero element, that means there are zero leading zeros.

We say a matrix is in *row echelon form* if each row has more leading zeros than the row before it. The number of leading zeros increases from row to row. Thus

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

is **not** in row-echelon form. The first row has two leading zeros, the second row has none, and the bottom row has one leading zero. The order of the rows needs to be changed to put the matrix in row-echelon form. In this case, we need to interchange rows one and two, and then interchange the new row two (old row one) with row three to put it in row-echelon form.

In contrast, the matrices

$$\mathbf{B} = \begin{pmatrix} 1 & 3/2 & 1/2 & 5 \\ 0 & 0 & -1/2 & -20 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

are in row-echelon form. Matrix $\mathbf{B}$ has no leading zeros in row one, and two leading zeros in row two. The rows are arranged so that the number of leading zeros is increasing, the hallmark of row-echelon form.

Matrix $\mathbf{C}$ has no leading zeros in the first row, one in the second row, and two leading zeros in the third row. Again, the number of leading zeros is increasing, so $\mathbf{C}$ is in row-echelon form.

7.12 Reduced Row-Echelon Form

A matrix is in *reduced row-echelon form* if (1) it is in row-echelon form, (2) each leading non-zero term is 1, and (3) there are no other non-zero terms in each column where there is a leading 1 (we have pivoted using each leading 1).

The matrix **B** above is in row-echelon form, but not reduced row-echelon form because the leading non-zero term in the second row is $-1/2$ rather than 1. Even if we change it to 1, it is still not in reduced row-echelon form because there is another non-zero term in the second column.

The matrix **C** is also not in reduced row-echelon form even though every row has leading one. The leading ones in rows one and two are in otherwise zero columns. The problem is in row three. The leading one is in column three and the rest of the column is not zero. Further row reduction is possible by pivoting on the entry in row three, column three.

Here is a matrix in reduced row-echelon form:

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Row one has one leading zero while row two has three leading zeros. Moreover, both columns two and four are pivot columns. Only row one has a non-zero entry in column two and only row two has a non-zero entry in column four.

7.13 Gauss-Jordan and Reduced Row-Echelon Form

The matrix

$$\begin{pmatrix} 1 & 0 & 4/11 & -5/11 \\ 0 & 1 & 1/11 & 40/11 \end{pmatrix},$$

which we found using the Gauss-Jordan procedure, is in reduced row-echelon form, as is

$$\mathbf{R} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Gauss-Jordan method always yields a matrix in reduced row-echelon form due to the pivot steps. The simpler method of Gaussian elimination, which omits the pivot step, puts the matrix in row-echelon form, but it will usually not be reduced row-echelon form. Gaussian elimination is more efficient for calculation, but Gauss-Jordan elimination is more useful for understanding the solution.

7.14 Gauss-Jordan always yields a Reduced Matrix

All matrices produced by the Gauss-Jordan procedure must be in reduced row-echelon form.

Theorem 7.14.1. *If* $\mathbf{A}$ *is a matrix produced by the Gauss-Jordan procedure, it is in reduced row-echelon form.*

Proof. In each cycle of the Gauss-Jordan procedure, first find a row with a non-zero term as far left as possible, then divide to make the leading term 1. Following that, we pivot to eliminate any non-zero terms in that column. At that point the rows below the pivot row have more leading zeros than the pivot. As we repeat the Gauss-Jordan cycle, the number of leading zeros increases.

When the Gauus-Jordan procedure is done, the higher the row, the farther left the leading 1 will be. Moreover, each leading 1 is in a column that is otherwise zero due to the pivot step. □

7.15 Basic and Free Variables

If an $m \times n$ matrix is in reduced row-echelon form, a variable is *basic* if its column contains a pivot, and *free* if its column contains no pivot. Notice that every variable is either basic or free, but never both.

In the reduced coefficient matrix

$$\mathbf{R} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

variables x_2 and x_5 have pivots, so they are basic. The other variables, (x_1, x_3, x_4), are free.

Because each of the n variables is either basic or free, but not both, we have the following equation.

$$\#\text{basic vars} + \#\text{free vars} = n. \tag{7.15.2}$$

7.16 Basic Variables, Free Variables, and Rank

As we saw earlier we can reduce the system

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 10 \\ 5x_1 + 2x_2 + 2x_3 &= 5 \end{aligned} \tag{7.7.1}$$

to

$$\begin{aligned} x_1 + \frac{4}{11}x_3 &= -\frac{5}{11} \\ x_2 + \frac{1}{11}x_3 &= \frac{40}{11}. \end{aligned}$$

When a system has a solution and one or more free variables, we can choose the values of the free variables (x_3 here) to be anything we wish. After the free variables have been chosen, the values of the basic variables are fully determined.

The set of solutions in equation (7.7.1) depends linearly on a single parameter, x_3. That means the solution set is a line, an infinite straight line. If we had two free variables, we would get a plane. Three free variables would give us a 3-dimensional solution set. This solution set contains all lines generated by any two distinct points in it, and all planes generated by three non-collinear points in it. The situation in higher dimensions is similar.

Because the number of free variables is tied to the solution set, it remains unchanged under any of the elementary equation/row operations.[1] Then we can use equation (7.15.2) to see that the number of basic variables is also unaffected by elementary row operations.

We now define the *rank* of an $m \times n$ matrix as the number of basic variables it possesses. The rank obeys

$$\operatorname{rank} \mathbf{A} + \#\text{free vars} = n.$$

Since the rank can be derived from number of variables and type of the solution set (point, line, plane, etc.), it remains unchanged whenever we use elementary row or equation operations.

[1] This is a argument in favor of the statement, but is not yet a proof. Chapter 27 of Simon and Blume has more on this. See section 27.4 of the notes.

7.17 The Rank of a Matrix

If a matrix is in row echelon form, the number of basic variables will be the number of non-zero rows in the matrix. It follows that the *rank* of a matrix can equivalently be defined as the number of non-zero rows in row echelon form. It does not depend on which row echelon form we use as they all have the same solution set and so the same number of basic and free variables. In fact, the solution set is a flat set Z in $\mathbb{R}^n$ with dimension equal to the number of free variables.[2] The rank is then

$$\operatorname{rank} \mathbf{A} = n - \dim Z = n - \#\text{free vars}.$$

Theorem 7.17.1. *The rank of a matrix has the following properties.*

1. *If* $\hat{\mathbf{A}}$ *is an augmented matrix, then either* $\operatorname{rank} \hat{\mathbf{A}} = \operatorname{rank} \mathbf{A}$ *or* $\operatorname{rank} \hat{\mathbf{A}} = 1 + \operatorname{rank} \mathbf{A}$.
2. $\operatorname{rank} \mathbf{A} \leq$ *number of rows of* $\mathbf{A}$.
3. $\operatorname{rank} \mathbf{A} \leq$ *number of columns of* $\mathbf{A}$.

Proof. When we row-reduce $\hat{\mathbf{A}}$, we obtain a matrix of the form $(\mathbf{A}^*|\mathbf{B}^*)$ where $\mathbf{A}^*$ is a row reduction of $\mathbf{A}$ and $\mathbf{B}^*$ is an $m \times 1$ matrix obtained by row reducing the $\mathbf{B}$ portion of the augmented matrix. Either $\mathbf{B}^*$ adds a non-zero row, or it does not, showing that $\operatorname{rank} \hat{\mathbf{A}}$ is either $\operatorname{rank} \mathbf{A}$ or $\operatorname{rank} \mathbf{A} + 1$.

Since the number of non-zero rows in the row reduction is no more than the number of rows in $\mathbf{A}$, (2) follows.

For (3), every non-zero row in the row reduction has a leading non-zero entry is in a different column, so there are no more leading non-zero entries than there are columns, establishing the result. □

[2] We will not show this until after we have defined dimension in Chapter 11.

7.18 Rank and Solutions

A key result relates the ranks of a matrix and of its augmented matrix to the existence of a solution to the linear system described by the augmented matrix.

Augmented Rank Theorem. *A linear system with coefficient matrix* $\mathbf{A}$ *and augmented matrix* $\hat{\mathbf{A}}$ *has a solution if and only if* $\operatorname{rank}\mathbf{A} = \operatorname{rank}\hat{\mathbf{A}}$.

Proof. As in the proof of Theorem (7.17.1), we can write the row-reduced matrix for $\hat{\mathbf{A}}$ in the form $(\mathbf{A}^*|\mathbf{B}^*)$.

If case: $\operatorname{rank}\mathbf{A} = \operatorname{rank}\hat{\mathbf{A}}$, then each non-zero row contains a basic variable and we can find a solution by setting all of the free variables to zero.

Only if case: If the system has a solution, then $\mathbf{B}^*$ cannot have a non-zero entry corresponding to a zero row of $\mathbf{A}^*$ as it would contradict the fact that we have a solution. It follows that $\operatorname{rank}\mathbf{A} = \operatorname{rank}\hat{\mathbf{A}}$. □

7.19 Corollaries to Augmented Rank Theorem

In these corollaries, $\mathbf{A}$ is an $m \times n$ coefficient matrix.

Corollary 7.19.1. *A system of linear equations either has no solution, one solution or infinitely many solutions.*

Proof. By the Augmented Rank Theorem, there is no solution if and only if $\operatorname{rank} \mathbf{A} < \operatorname{rank} \hat{\mathbf{A}}$. Furthermore, there will be solutions if and only $\operatorname{rank} \mathbf{A} = \operatorname{rank} \hat{\mathbf{A}}$. In this case, if there are any free variables there will be infinitely many solutions. That leaves the case of no free variables where the solution is unique. $\square$

Corollary 7.19.2. *If a linear system has a unique solution, then $m \geq n$, meaning it has at least as many equations as variables.*

Proof. If there is a unique solution, there are no free variables. Every unknown must be basic, and that requires at least n equations, so $m \geq n$. $\square$

Corollary 7.19.3. *If a linear system has more variables than its rank, either it has no solution or infinitely many solutions. In particular, this holds if it has more variables than equations, if $n \geq m$.*

Proof. We know that for an $m \times n$ matrix $\mathbf{A}$, $n = \operatorname{rank} \mathbf{A} + \#\text{free vars}$ since the rank is the number of basic variables. If $n > \operatorname{rank} \mathbf{A}$, there must be free variables and so either no solution, or infinitely many solutions.

Since $\operatorname{rank} \mathbf{A} \leq m$, the number of equations, the same holds if there are more variables than equations. $\square$

It is possible to not have a solution in Corollary 7.19.3. Consider the case

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= 1 \\ 3x_1 + 0x_2 + 0x_3 &= 12 \end{aligned}$$

which has no solution even thought there are more variables (3) than equations (2).

7.20 Homogeneous Linear Systems

We say a system of linear equations is *homogeneous* if the right-hand side of every equation is zero. That is, it has form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= 0 \\
\vdots \qquad\qquad\qquad & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned} \tag{7.20.3}$$

7.21 Solution of Homogeneous Systems

One nice thing about homogeneous linear systems is that they always have a solution.

We can see that in general terms. Since the right-hand side of a homogeneous system is zero, the constant terms cannot affect the rank of the augmented matrix and $\operatorname{rank} \mathbf{A} = \operatorname{rank} \hat{\mathbf{A}}$. There will be at least one solution, leaving only the question is whether there are many solutions or just one.

Or we can find a universal solution for homogeneous systems, as in the following theorem.

Theorem 7.21.1. *Let* $\mathbf{A}$ *be the* $m \times n$ *coefficient matrix of a homogeneous linear system. The zero vector* $(x_1, \ldots, x_n) = (0, \ldots, 0)$ *solves this system.*

Proof. If you plug the zero value into equation (7.20.3), you will see they solve the system. □

Now that we know the system can be solved by the zero vector, we still need to see whether that is the only solution.

Corollary 7.21.2. *A homogeneous linear system has infinitely many solutions if and only if there are more unknowns than the rank of the* $m \times n$ *coefficient matrix, if* $n > \operatorname{rank} \mathbf{A}$. *Equivalently, the solution* $(0, \ldots, 0)$ *is unique if and only if* $n = \operatorname{rank} \mathbf{A}$.

Proof. **If case:** This part is Corollary 7.19.3, taking account of the fact that our system has a solution.

Only if case: Suppose the system has infinitely many solutions. This means there are free variables, so $n > \operatorname{rank} \mathbf{A}$.

For the other part, note that $n \leq \operatorname{rank} \mathbf{A}$, so either $n = \operatorname{rank} \mathbf{A}$ or $n > \operatorname{rank} \mathbf{A}$. The latter case is equivalent to having infinitely many solutions, so the form is equivalent to having a unique solution. □

7.22 Properties of Linear Systems I

We now take an $m \times n$ coefficient matrix $\mathbf{A}$ and consider the various systems of linear equations that may be formed that use this coefficient matrix. We examine what happens when we alter the right-hand side of the system (i.e., $(b_1, \ldots, b_n)$).

We list several corollaries to our previous results. In the proofs below, we will use n for the number of variables and m for the number of equations.

Corollary 7.22.1. *Let $\mathbf{A}$ be a matrix of coefficients for a linear system. The rank of $\mathbf{A}$ is equal to the number of rows of $\mathbf{A}$ if and only if the system has a solution for every right-hand side.*

Proof. If rank $\mathbf{A}$ is the number of rows of $\mathbf{A}$, it follows that rank $\hat{\mathbf{A}}$ is also the number of rows of $\mathbf{A}$ regardless of the right-hand side. Then the system has a solution by the Augmented Rank Theorem.

Now suppose the system has a solution for every right-hand side. If the rank of $\mathbf{A}$ where less than the number of rows, we may row reduce the matrix to obtain $\mathbf{A}^*$, which will have at least one zero row. Now let $(b_1, \ldots, b_n) = (0, \ldots, 0, 1)$. This of course has no solution. We may now reverse the row reduction on the augmented matrix $\hat{\mathbf{A}}^*$ to obtain a system with coefficient matrix $\mathbf{A}$ and right-hand side obtained by undoing the row reduction that has no solution, contradicting the fact that original the system has a solution for every right-hand side. This contradiction shows that the rank of $\mathbf{A}$ must equal the number of rows. $\square$

7.23 Properties of Linear Systems II

Corollary 7.23.1. *If a system of linear equations has more equations than unknowns there is a right-hand side where there are no solutions.*

Proof. If there are more equations than unknowns, rank $\mathbf{A} < m$. We can row-reduce $\mathbf{A}$ to obtain a matrix $\mathbf{A}^*$ with one or more zero rows. Now let $(b_1, \ldots, b_n) = (0, \ldots, 0, 1)$. This of course has no solution. We may now reverse the row reduction on the augmented matrix $\hat{\mathbf{A}}^*$ to obtain a system with coefficient matrix $\mathbf{A}$ and right-hand side obtained by undoing the row reduction that has no solution. □

Corollary 7.23.2. *A system of linear equations has at most one solution for every right-hand side if and only if the rank of the coefficient matrix $\mathbf{A}$ is equal to the number of columns of $\mathbf{A}$.*

Proof. Suppose the rank of the coefficient matrix is equal to the number of columns. In this case there are no free variables, and if a solution exists, it must be unique. □

7.24 Singular and Non-singular Matrices

We say a matrix $\mathbf{A}$ of coefficients is *non-singular* if there exists a unique solution to the system for every right-hand side. Conversely, a coefficient matrix $\mathbf{A}$ is *singular* if there is either a right-hand side with no solution or a right-hand side with infinitely many solutions. In the latter case, the homogeneous system will also have infinitely many solutions.

Corollary 7.24.1. *An $m \times n$ matrix $\mathbf{A}$ is non-singular if and only if the number of rows and columns of $\mathbf{A}$ are both equal to* $\operatorname{rank}\mathbf{A}$*, that is* $m = n = \operatorname{rank}\mathbf{A}$.

Proof. By Corollary 7.22.1, a matrix $\mathbf{A}$ has a solution for every right-hand side if and only if $\operatorname{rank}\mathbf{A}$ is equal to the number of rows of $\mathbf{A}$. By Corollary 7.23.2, $\mathbf{A}$ has at most one solution for every right-hand side if and only if $\operatorname{rank}\mathbf{A}$ is the number of columns of A.

These two conditions together characterize non-singular matrices, proving the result. $\square$

7.25 Reduction of Non-singular Matrices

If a matrix is non-singular, its reduced row-echelon form is special.

Lemma 7.25.1. *Suppose an $n \times n$ matrix $\mathbf{A}$ is non-singular. Then its reduced row-echelon form is*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{7.25.4}$$

In other words, it has 1*'s on the diagonal and* 0*'s everywhere else.*

Proof. We know that $\mathbf{A}$ is square by Corollary 7.24.1. With n columns and n non-zero rows, the only way to write a matrix in reduced row-echelon form is as in equation (7.25.4).

The matrix in equation (7.25.4) is referred to as the *identity matrix*. We will be properly introduced to it in section 8.12.

7.26 Linear Implicit Function Theorem

Linear Implicit Function Theorem. *Consider a system with coefficient matrix* $\mathbf{A}$ *and constant terms* $\mathbf{B}$. *Suppose the variables are partitioned into* k *endogenous variables,* $(x_1, \dots, x_k)$ *and* $n - k$ *exogenous variables* $(x_{k+1}, \dots, x_n)$. *Then for every choice of the exogenous variables* $(x_k^0 + 1, \dots, x_n^0)$ *and constant terms* $\mathbf{B}$*, there exist unique values of the endogenous variables* $(x_1^0, \dots, x_k^0)$ *so that* $(x_1^0, \dots, x_n^0)$ *solve the system if and only if*

(a) $k = m$*, i.e., the number of endogenous variables, is the number of rows of* $\mathbf{A}$.

(b) $\operatorname{rank} \mathbf{A}_1 = k$*, where* $\mathbf{A}_1$ *is the matrix formed by the first* k *columns and rows of* $\mathbf{A}$.

Proof. If case: Assuming (a) and (b), consider the reduced row-echelon form of $\mathbf{A}$. Corollary (7.24.1) tells us that $\mathbf{A}_1$ is non-singular, so in the reduced row-echelon form it is given by equation (7.25.4). That means that we write the basic variables $x_1, \dots, x_k$ in terms of the free variables $x_{k+1}, \dots, x_n$.

Only if case: Since we can always solve for the k endogenous variables, Corollary 7.23.1 tells us that there are at most k equations (and hence k rows in $\mathbf{A}$). This establishes (a).

We can now write the coefficient matrix as $(\mathbf{A}_1|\mathbf{A}_2)$ where $\mathbf{A}_1$ has k rows and k columns. The other variables are being treated as constants, and for this to always have a unique solution, the rank of $\mathbf{A}_1$ must be k. Thus (b) holds. □

This theorem allows us to write $(x_1, \dots, x_k)$ as a function of $(x_{k+1}, \dots, x_n)$.

You'll notice that compared to the text, I added the requirement that there is a solution for every set of constant terms $\mathbf{B}$. Without that, the theorem can fail if the exogenous variables have no effect on the equations (i.e., their coefficients are all zero).

7.27 Failure of the Uncorrected Theorem

Consider the following system

$$\begin{aligned} x_1 + x_3 + x_4 &= 1 \\ 0x_2 + 0x_3 + 0x_4 &= 0. \end{aligned}$$

Consider the case where there is one endogenous variable (x_1) and x_2, x_3, x_4 are all exogenous. For every choice of the endogenous variables, there is a unique value of $x_1 = 1 - x_3 - x_4$ that solves the system. Nonetheless, the theorem fails as $m = 2 > 1 = k$.

By requiring a unique solution for all right-hand sides and endogenous variables, we eliminate this possibility. This system fails that condition does not have a solution when $b_2 \neq 0$.

Yes, its pretty contrived, but the statement of the theorem in Simon and Blume allows it. My version rules out this case.

7.28 Example I: Linear Implicit Function Theorem

Our first example is a case where it works as planned. We let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 5 \\ 1 & 3 & 2 & 12 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 2 & 1 & 5 & 1 \\ 1 & 3 & 2 & 12 & 0 \end{pmatrix}$$

which row reduces to

$$\begin{pmatrix} 1 & 0 & -1 & -9 & 3 \\ 0 & 1 & 1 & 7 & -1 \end{pmatrix}$$

The corresponding equations are

$$\begin{aligned} x_1 &= 3 + x_3 + 9x_4 \\ x_2 &= -1 - x_3 - 7x_4 \end{aligned}$$

and it is evident that for every choice of the exogenous variables (x_3, x_4), we get unique values of the endogenous variables (x_1, x_2).

7.29 Example II: Linear Implicit Function Theorem

The Linear Implicit Function fails in the following case, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The augmented matrix is

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 \end{pmatrix}$$

which row reduces to

$$\begin{pmatrix} 1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$

with x_1 and x_2 intended as endogenous variables. However, the only equation involving x_1 and x_2 is

$$x_1 + x_2 = 2 - 3x_4.$$

While this certainly has solutions for any values of exogenous variables x_3 and x_4, the solution fails to be unique. The problem here is that with $k = 2$ endogenous variables,

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

has rank one, which is less than $k = 2$.

Here we must reconsider our choice of exogenous and endogenous variables.

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