

## 33. How to Count

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There are few mathematical concepts more fundamental than counting. Because most of us have known how to count since an early age, we rarely give it much thought. Nonetheless, it pays to examine counting closely as there are complications that become apparent when numbers become very large.

### 33.1 Counting

We start by re-learning how to count.

One way to count the elements of a set is to put them in a one-to-one correspondence with the elements of another set. Young children often do this using their fingers. They associate each finger with one of the objects they are counting, which works fine up to ten. We can do the same sort of thing by using a bijective (one-to-one and onto) mapping.

We formalize this by saying that  $A$  and  $B$  have the same *cardinality*, denoted  $\#A = \#B$ , if there is a bijective correspondence between them, a one-to-one function mapping  $A$  onto  $B$ . The concept of cardinality formalizes what it means for sets to have the same number of number of elements.<sup>1</sup>

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<sup>1</sup> If we want to consider the order of items—first, second, third—we would need ordinal numbers rather than cardinal numbers. Halmos, *Naive Set Theory* (1960) has a nice introduction to both cardinal and ordinal arithmetic.

### 33.2 Matching Up Sets

The sets  $A = \{1, 2, 3\}$  and  $B = \{\text{red, white, blue}\}$  each contain 3 elements. Both have the same cardinality. One mapping that establishes this is  $f(1) = \text{red}$ ,  $f(2) = \text{white}$ ,  $f(3) = \text{blue}$ . This is not the only mapping that shows these sets have the same cardinality. Another that works is  $g(1) = \text{white}$ ,  $g(2) = \text{red}$ ,  $g(3) = \text{blue}$ . We could even use a mapping in the opposite direction, such as  $h(\text{red}) = 3$ ,  $h(\text{white}) = 1$ , and  $h(\text{blue}) = 2$ . As you can, we can easily create a bijective correspondence between any set with three elements and  $\{1, 2, 3\}$ .

Instead of using their fingers, adults count using a mental reference set, the *natural numbers*  $\mathbb{N} = \{1, 2, 3, \dots\}$ .<sup>2</sup> We count by mentally putting the set to be counted into bijective correspondence with an initial segment of  $\mathbb{N}$ . Thus if we count  $A$  and find it has  $n$  elements, we have put it into a bijective correspondence with  $\{1, 2, 3, \dots, n\}$ . We can define  $n = \#\{1, \dots, n\}$ .

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<sup>2</sup> The names “counting numbers” and “natural numbers” are not standardized and definitions vary depending on whether or not zero is included. Simon and Blume use  $\mathbb{N}$  for the natural numbers, and I follow that so you will have a consistent notation. It apparently took a long time before the number 0 was discovered, so it doesn’t get called “natural”. Egyptian accountants were using a special symbol for a zero balance as early as 1770 BC. The first recorded use of zero as a proper number was by the Indian mathematician Bramagupta (c. 598–c. 668). Bramagupta wrote about the various properties of zero. There’s some evidence it was used as part of their decimal system rather earlier.

### 33.3 Equivalence Relations

We won't have to worry about counting producing inconsistent results because having equal cardinality is an equivalence relation.

What does that mean?

**Equivalence Relation.** A binary relation  $\sim$  on a set  $X$  is an *equivalence relation* if:

1. It is *reflexive*.  $x \sim x$ .
2. It is *symmetric*. If  $x \sim y$  then  $y \sim x$ .
3. It is *transitive*. If  $x \sim y$  and  $y \sim z$ , then  $x \sim z$ .

### 33.4 Cardinality is an Equivalence Relation

We use this concept by defining sets to be equivalent if they have the same cardinality. Cardinality, which measures the number of elements in a set, is an equivalence relation. Then any two sets with the same number of elements, say will match up with every other set with the same number of elements.

**Theorem 33.4.1.** *For sets  $A$  and  $B$ , define  $A \sim B$  if and only if  $\#A = \#B$ . Then  $\sim$  is an equivalence relation.*

**Proof.** We have to show that all three properties are satisfied. Unpacking the definition,  $A \sim B$  means there is a bijective mapping  $f: A \rightarrow B$ .

(1)  $\sim$  is reflexive. Define  $f: A \rightarrow A$  by  $f(x) = x$ . Then  $f$  is a bijective mapping from  $A$  to  $A$ , so  $A \sim A$ .

(2)  $\sim$  is symmetric. If  $A \sim B$ , there is a bijective mapping  $f: A \rightarrow B$ . Since  $f$  is bijective, the unique inverse mapping,  $f^{-1}$ , is a bijective mapping  $f^{-1}: B \rightarrow A$ . This shows that  $B \sim A$ , proving symmetry.

(3)  $\sim$  is transitive. Suppose we have three sets, with  $\#A = \#B$  and  $\#B = \#C$ . Then there are bijective mappings  $f: A \rightarrow B$  and  $g: B \rightarrow C$ . Define  $h: A \rightarrow C$  by  $h = g \circ f$ .

$$A \xrightarrow{f} B \xrightarrow{g} C$$

The composition of bijective maps is bijective, so  $\#A = \#C$ , establishing transitivity.

It follows that  $\sim$  is an equivalence relation. ■

### 33.5 Larger and Smaller Sets

We can also define smaller and larger cardinality using one-to-one correspondences. A set  $A$  is *no bigger than* set  $B$  if there is a one-to-one mapping of  $A$  **into**  $B$ . In symbols, we write  $\#A \leq \#B$ . In that case, there is a one-to-one mapping between  $A$  and  $f(A)$ , a subset of  $B$ . If  $A$  is no bigger than  $B$  and  $B$  is no bigger than  $A$ , they have the same cardinality, even if the sets are infinite. This is a consequence of the Schroeder-Bernstein Theorem, which holds for sets of any size.<sup>3</sup>

**Schroeder-Bernstein Theorem.** *Let  $A$  and  $B$  be sets such that  $A$  is no bigger than  $B$  and  $B$  is no bigger than  $A$ , then there is a one-to-one mapping of  $A$  onto  $B$ . In other words,  $A$  and  $B$  have the same cardinality.*

**Proof.** See pg. 88 of Halmos (1960). ■

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<sup>3</sup> Felix Bernstein (1878–1956) was a German mathematician. He worked on related problems, and also discovered how inheritance of blood types works.

Ernst Schroeder (1841–1902) was also a German mathematician. He worked on mathematical logic, and his "Lectures on the Algebra of Logic, 1890–1905" systematized the systems of formal logic then known.

### 33.6 Cardinality of Infinite Proper Subsets

When dealing with finite sets, proper subsets have fewer elements and so a smaller cardinality. That is no longer true when sets are infinite. Proper subsets of infinite sets can even have the same cardinality as the original!

Consider the natural numbers  $\mathbb{N} = \{1, 2, 3, \dots\}$  and non-negative integers  $\mathbb{Z}_0 = \{0, 1, 2, \dots\}$ .

Even though there is clearly an extra element in  $\mathbb{Z}_0$ , both  $\mathbb{Z}_0$  and  $\mathbb{N}$  have identical cardinality. Here  $f(n) = n + 1$  is the desired one-to-one mapping from  $\mathbb{Z}_0$  onto  $\mathbb{N}$ . A similar mapping has some importance in monetary theory in Example 33.8.1.

### 33.7 Cardinality of Even and Odd Numbers

Two examples of infinite proper subsets with the same cardinality as the original set start with the positive integers. These are the sets of the even positive integers and of the odd positive integers.

► **Example 33.7.1: Cardinality of Sets of Odd and Even Numbers.** The sets of positive odd numbers and of positive even numbers each have the same cardinality as the set of natural numbers  $\mathbb{N}$ . Consider the following mapping defined for each natural number,  $f(n) = 2n$ . As a mapping from the positive integers to the even positive integers, this is clearly one-to-one and maps the natural numbers onto the positive even numbers. The mapping  $g(n) = 2n - 1$  accomplishes the same thing for the positive odd numbers, bijectively mapping  $\mathbb{N}$  onto them. It follows that all three sets have the same cardinality, denoted  $\aleph_0$ . ◀

### 33.8 Overlapping Generations and Cardinality

► **Example 33.8.1: Fiat Money.** Consider the following overlapping generations model. One agent (generation) is born at each time  $t$ , and lives two periods. Each agent earns 1 unit of a consumption good when young, and none when old. There is only one generation alive at time zero, two at all other times. Each person consumes  $c_y^t$  when young and  $c_o^t$  when old, obtaining utility  $u_t(c_y^t, c_o^t) = c_y^t + c_o^t$ .

Given prices  $p_t = 1$ , consuming the endowment maximizes utility. Since markets clear, we have an equilibrium without trade.

However, if the young generation at each time  $t > 0$  gives  $1 - 2^{-t}$  units of its endowment to the old generation, everyone is made better off since the generation young at time  $t$  gains  $1 - 2^{-(t+1)}$  when old at a cost of  $1 - 2^{-t}$  units when young. This yields a utility gain of  $2^{-t-1}$  for generation  $t$ . The transfers are  $(1/2, 3/4, 7/8, 15/16, \dots)$  starting at time  $t = 1$ .

This improvement cannot be an equilibrium. However, we can make it an equilibrium allocation by introducing a fiat currency. Give 1 unit of currency to the generation that is old at time zero. If goods have price one in each time period, and money has price  $1/2$  at  $t = 1$ , and appreciates every period according to the schedule  $(1/2, 3/4, 7/8, \dots)$ , the old generation can pay for the transfer with cash. The young generation takes the cash, and spends it in the next period to buy a bit more consumption when old (due to the appreciation of the currency). The desired pattern of transfers is now an equilibrium.

This is not the only monetary equilibrium, nor is it the only one that increases everyone's utility. ◀

### 33.9 Degrees of Infinity

Defining cardinality via one-to-one correspondences allows us to distinguish different degrees of infinity, even though some infinities have the same cardinality. We say an infinite set is *countably infinite* if it has the same cardinality as the natural numbers  $\mathbb{N} = \{1, 2, 3, \dots\}$ . We denote the cardinality of  $\mathbb{N}$  by  $\aleph_0$ . We call a set *countable* if it is either finite or countably infinite. Sets that are not countable are called *uncountable*. Uncountable sets are necessarily infinite.

We will use the following lemma to show that certain sets are countable.

**Lemma 33.9.1.** *Let  $A \subset \mathbb{N}$ . If there is a mapping  $f$  from  $A$  onto a set  $B$ , then  $B$  is countable*

**Proof.** If  $B$  is empty we may take  $A$  to be empty and there is nothing to prove.

If  $B$  is non-empty,  $A$  must also be non-empty. For each  $b \in B$  let  $g(b) = \min\{a \in A : f(a) = b\}$ . Since  $f$  is onto, the set will be non-empty and will have a minimum, so  $g(b)$  exists. Then  $g$  maps  $B$  onto a subset of  $A$  so  $\#B \leq \#A$ .

The fact that we have a mapping from  $A$  onto  $B$  shows  $\#B \leq \#A$ . By the Schroeder-Bernstein Theorem,  $\#A = \#B$ . ■

**33.10 A Bijection Between an Interval and the Real Line**

We will show that there is a bijection between the open interval  $(0, 1)$  and the entire real line  $\mathbb{R}$ . Moreover, this bijection that maps rational numbers in  $(0, 1)$  onto the rational numbers in  $\mathbb{R}$ .

**Lemma 33.10.1.** Define  $f: (0, 1) \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 1/(4x - 1) & \text{for } 0 < x < 1/4, \\ 4x - 2 & \text{for } 1/4 \leq x \leq 3/4, \\ 1/(4x - 3) & \text{for } 3/4 < x < 1. \end{cases}$$

Then

- (a) The function  $f$  maps  $(0, 1)$  onto  $\mathbb{R}$ .
- (b) If  $r \in \mathbb{Q}$  is rational, then  $f(r) \in \mathbb{Q}$ .
- (c) If  $f(x) \in \mathbb{Q}$ , then  $x \in \mathbb{Q}$ .

Moreover,  $f$  is a one-to-one mapping of the rational numbers in  $(0, 1)$  onto all of the rational numbers.

**33.11 Proof of Lemma 33.10.1**

**Proof.** We first prove each of the three conclusions.

Recall that  $f: (0, 1) \rightarrow \mathbb{R}$  is

$$f(x) = \begin{cases} 1/(4x - 1) & \text{for } 0 < x < 1/4, \\ 4x - 2 & \text{for } 1/4 \leq x \leq 3/4, \\ 1/(4x - 3) & \text{for } 3/4 < x < 1. \end{cases}$$

- (a) When  $0 < x < 1/4$ ,  $-1 < 4x - 1 < 0$ , so  $f(x) \in (-\infty, -1)$ . When  $1/4 \leq x \leq 3/4$ ,  $-1 \leq 4x - 2 \leq 1$ , so  $f(x) \in [-1, +1]$ . When  $3/4 < x < 1$ ,  $0 < 4x - 3 < 1$ , so  $f(x) \in (1, +\infty)$ . The inverse function for  $f$  is

$$f^{-1}(x) = \begin{cases} 1/4 + 1/4x & \text{for } -\infty < x < -1, \\ (x + 2)/4 & \text{for } -1 \leq x \leq +1, \\ 3/4 + 1/4x & \text{for } +1 < x < +\infty \end{cases}$$

Its existence shows that  $f$  is a bijective mapping.

- (b) Since  $f$  is defined by rational operations,  $f(r)$  will be rational whenever  $r$  is rational.
- (c) The inverse function  $f^{-1}$  is also defined by rational operations, so  $f^{-1}(r)$  is rational whenever  $r$  is rational. Thus any  $r \in \mathbb{Q}$  is in the image of  $f$ . In other words,  $f$  maps onto  $\mathbb{Q}$ .

Because (a)  $f$  maps onto  $\mathbb{R}$ , and both (b) maps rational numbers to rational numbers, and (c) every rational number is the image of a rational number, we can conclude that  $f$  is a one-to-one mapping of the rational numbers between  $(0, 1)$  onto all of the rational numbers. ■

### 33.12 The Rational Numbers are Countable

One important countably infinite set is the set of rational numbers,  $\mathbb{Q} = \{p/q : p, q \text{ are integers with } q \neq 0\}$ .

**Proposition 33.12.1.** *The set of rational numbers is countable.*

**Proof.** Since the natural numbers are a subset of the rational numbers, the Schroeder-Bernstein Theorem tells us it is enough to find a mapping of the natural numbers onto the rational numbers.

We employ the mapping from Lemma 33.10.1.

$$f(x) = \begin{cases} 1/(4x - 1) & \text{for } 0 < x < 1/4, \\ 4x - 2 & \text{for } 1/4 \leq x \leq 3/4, \\ 1/(4x - 3) & \text{for } 3/4 < x < 1. \end{cases}$$

By Lemma 33.10.1,  $f(x)$  is a one-to-one mapping of the rational numbers between 0 and 1 onto all of the rational numbers. It follows that we need only consider the rational numbers between 0 and 1. Consider the following array of rational numbers:

$$\begin{array}{cccccc} 1/2 & 1/3 & 1/4 & 1/5 & \dots & \\ & 2/3 & 2/4 & 2/5 & \dots & \\ & & 3/4 & 3/5 & \dots & \\ & & & 4/5 & \dots & \end{array}$$

All of the rational numbers in  $(0, 1)$  are listed. The number  $p/q$  is found in row  $p$ , column  $q$  as well as other locations such as row  $2p$ , column  $2q$ . Now define a mapping  $g(1) = 1/2$ ,  $g(2) = 1/3$ ,  $g(3) = 2/3$ ,  $g(4) = 1/4$ ,  $g(5) = 2/4$ ,  $g(6) = 3/4$ ,  $g(7) = 1/5$ , etc. This maps  $\mathbb{N}$  onto the rational numbers between 0 and 1. It follows that  $f \circ g$  maps  $\mathbb{N}$  onto  $\mathbb{Q}$ . This is our required mapping of  $\mathbb{N}$  onto  $\mathbb{Q}$ . ■

**33.13 Countability of Countable Unions**

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We can also show that countable unions of countable sets are countable.

**Lemma 33.13.1.** *A countable union of countable sets is countable.*

**Proof.** Let  $A_i$  be countable for each  $i$  and  $A = \cup_{i \in I} A_i$  where  $I$  is a countable set. Since each  $A_i$  is countable, we can label the elements of  $A_i$  so that  $A_i = \{a_1^i, a_2^i, \dots\}$ . We can also label the elements of  $I$  with  $I = \{i_1, i_2, \dots\}$ . Note that  $A_i$  or  $I$  may have only finitely many elements.

Define a function  $g: \mathbb{Q} \rightarrow \cup_i A_i$  by

$$g(r) = \begin{cases} a_p^{i_q} & \text{when } r = p + 1/q \text{ for } p, q \in \mathbb{N} \\ a_1^1 & \text{otherwise.} \end{cases}$$

This clearly maps  $\mathbb{Q}$  onto  $\cup_i A_i$ , showing that  $\cup_i A_i$  is countable. ■

One consequence is that the set of open intervals with rational endpoints is itself countable. For each rational number  $q$ , consider the set of open intervals starting at  $q$ ,

$$A_q = \{(q, r) : r \in \mathbb{Q}\}.$$

Each  $A_q$  is countable, and the set of rational intervals is the countable union  $\cup_{q \in \mathbb{Q}} A_q$ , which is countable by Lemma 33.13.1.

This set of rational intervals seems pretty big, but is still countable, no bigger than the set of natural numbers. This tells that uncountable sets have to be really big!

### 33.14 The Real Numbers are Uncountable!

We know such a set. The real numbers are **not countable**. We show that using *Cantor's Diagonal Argument* in the following example.<sup>4</sup>

► **Example 33.14.1: The Real Numbers are Uncountable.** In fact, we will show that the unit interval is uncountable. Since it is a subset of the real numbers, the real numbers must also form an uncountable set.

Suppose  $[0, 1]$  is countably infinite. Then we can put it into one-to-one correspondence with the natural numbers. This allows us to list the elements of  $[0, 1]$  in the order of their correspondence with 1, 2, 3, etc.

Write the real numbers on the list in decimal form. The list will look something like this:

$$\begin{array}{l} 0.182083820\dots \\ 0.398020208\dots \\ 0.972832711\dots \\ 0.002820337\dots \\ 0.137992365\dots \\ \vdots \end{array}$$

We can now use an argument due to Georg Cantor (Cantor, 1891) to find a real number in the interval  $[0, 1]$  that is not on the list. Take the  $n^{\text{th}}$  digit of the  $n^{\text{th}}$  number on the list. If this digit is less than 7, change it to 8. If it is more than 7, change it to 5. The diagonal has been shown in red above. In our case, it is 0.19289... which we transform to 0.85855.... This number differs in digit  $n$  from the  $n^{\text{th}}$  number on the list, so it cannot itself be on the list.<sup>5</sup> This contradicts the fact that all real numbers in  $[0, 1]$  are on the list and shows that the real numbers are not countable. ◀

The argument in Lemma 33.13.1 can be adapted to show that the open interval  $(0, 1)$  has the same cardinality as the entire real line.

<sup>4</sup> Georg Cantor (1845–1918) was one of the leading German mathematicians of his era. His theory of transfinite cardinal and ordinal numbers revolutionized the way mathematicians looked at infinite sets. His work helped establish not only the new foundations of analysis, but of all of mathematics.

<sup>5</sup> Notice that the way we changed the digits avoids problems with numbers that can be written two ways, such as 1 and 0.999....

### 33.15 The Cardinality of $\mathbb{R}$

The cardinality of  $\mathbb{R}$  is denoted  $\mathfrak{c}$  and referred to as the *cardinality of the continuum*. We now know that  $\mathfrak{c} > \aleph_0$ . You may wonder whether there are any sets that are larger than the rational numbers but smaller than the reals.

This is a deep question!

For many years, the answer was unknown. In 1963, Paul Cohen found a remarkable result. This question doesn't have a definitive answer! The question is undecidable. There are consistent mathematical systems, in which the axioms of set theory (and so all usual mathematics) hold, where there are such sets. Equally, there are other systems, consistent with normal mathematics, where such sets do not exist. Cohen showed how to construct such mathematical systems (Cohen, 1963).<sup>6</sup>

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<sup>6</sup> Paul Cohen (1934–2007) was an American mathematician who is best known for showing that the continuum hypothesis, that there are no cardinal numbers between  $\aleph_0$  and  $\mathfrak{c}$ , is independent of Zermelo-Fraenkel (ZF) set theory together with the Axiom of Choice (AC). Cantor believed the continuum hypothesis was true, but was unable to prove it. Well, Cohen showed it wasn't false, but wasn't true either. He developed a mathematical method called "forcing" that allowed him to show that there were systems of mathematics that were consistent with ZF and AC, where the continuum hypothesis was true, and that there were other such systems where the continuum hypothesis was false. In other words, the continuum hypothesis could neither be proved nor disproved solely by using ZF + AC.

## **12. Limits and Open Sets**

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This chapter draws on Chapter 12 of Simon and Blume. It focuses on limits, open sets, closed sets, and related concepts. In other words, it's about topology.

A large chunk of topology is about points being either near each other or far away. This holds true in normed and metric spaces where we have a precise notion of nearness, or more general spaces where nearness is not a numerical concept.

There are several ways to approach topology. One is to start with sequences and metric spaces. Then use the metric to define open balls and open sets, and convergent sequences to define closed sets.

Another is to start with the open sets themselves. It has the advantage of generality, but dodges the question of how we recognize a open set. To be sure, there are ways to answer the question: topological bases and subbases, neighborhood bases, nets, filters, etc. But the constructions are both more involved and more abstract than using a metric.

We will build things up via the more concrete approach of metric spaces and sequences, and discuss more general topologies to see the overall structure.

## 12.1 Sequences

A sequence of vectors in  $\mathbb{R}^m$  looks like this:<sup>1</sup>

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots\}$$

This sequence maps  $n$  to  $\mathbf{x}_n$ .

**Sequence.** A sequence in a set  $A$  is a mapping  $n \mapsto x_n$  from the natural numbers  $\mathbb{N}$  into  $A$ .<sup>2</sup>

We usually indicate a sequence by  $\{x_n\}_{n=1}^{\infty}$  or simply  $\{x_n\}$ . Sometimes we will write sequences using superscripts,  $\{x^n\}$ , particularly when writing sequences of vectors. We prefer not to write it as  $x(n)$ . This is partly because it helpful to have a distinct notation for this particular type of function, and because such notation could lead to confusion if the elements of the sequence are functions themselves.

A sequence may be described by a formula, such as

$$x_n = n^2 + \frac{1}{n},$$

or by otherwise describing the sequence.

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<sup>1</sup> We use  $\mathbb{R}^m$  here instead of  $\mathbb{R}^n$  because we will be using  $n$  in sequences.

<sup>2</sup> Some prefer to number sequence elements  $x_0, x_1, \dots$ . It works the same either way.

## 12.2 Examples of Sequences

Examples of sequences include

► **Example 12.2.1: Examples of Real-valued Sequences.**

$$\begin{aligned}x_n &= n && \{1, 2, 3, 4, \dots\}, \\x_n &= 1/n && \{1, 1/2, 1/3, 1/4, \dots\}, \\x_n &= (-1)^{n+1} && \{+1, -1, +1, -1, \dots\}, \text{ and} \\x_n &= (-1)^{n+1}n^2 && \{1, -4, 9, -16, 25, -36, \dots\}.\end{aligned}$$

All of the above examples are sequences. They are mappings from  $\mathbb{N}$  to  $\mathbb{R}$ . ◀

The examples above show that the long-run behavior of the sequence can vary. The first continually increases without bound. The second decreases toward zero, the third bounces back and forth between two values, and the fourth one makes bigger and bigger oscillations as  $n$  increases.

### 12.3 Convergent Sequences in $\mathbb{R}$

One type of sequence that plays an important role in mathematics is a convergent sequence. We start by considering convergent sequences of real numbers. After a little practice, we will graduate to  $\mathbb{R}^m$  and beyond.

**Convergent Sequences in  $\mathbb{R}$ .** We say that  $x_n$  *converges to*  $x$ , written

$$x_n \rightarrow x, \quad \text{or} \quad \lim_{n \rightarrow \infty} x_n = x,$$

if for every  $\varepsilon > 0$ , there is a positive integer  $N$  with  $|x_n - x| < \varepsilon$  for every  $n \geq N$ . A sequence is *convergent* if it converges to some  $x$ .

The definition means that no matter what we pick as a standard for closeness to  $x$  (call it  $\varepsilon$ ), the rest of the sequence will eventually meet that standard. The *tails* of the sequence, the points  $\{x_n\}_{n=N}^{\infty}$ , must always be close to the limit. They must eventually stay within some  $\varepsilon$  distance of the limit  $x$ .

Here “eventually” means when  $n$  is large enough, when  $n \geq N$  for some  $N$ . Convergent sequences can’t just occasionally visit a point, they have to settle down near it, within every distance  $\varepsilon > 0$ .

**12.4 The Sequence**  $\{1/n\}$ 

► **Example 12.4.1: The Sequence  $\{1/n\}$  Converges.** Let's see how this works with the sequence  $x_n = 1/n$ . I claim this sequence converges to  $x = 0$ . To prove it, we take any  $\varepsilon > 0$ . We must find an  $N$  so that

$$|1/n - 0| = 1/n < \varepsilon$$

for  $n \geq N$ . To get  $1/n < \varepsilon$ , we need  $n > 1/\varepsilon$ . So we take  $N$  with  $N > 1/\varepsilon$ . Then

$$n \geq N > \frac{1}{\varepsilon} \Rightarrow x_n = \frac{1}{n} < \frac{1}{N} < \varepsilon,$$

so

$$|x_n - 0| = |1/n - 0| = |1/n| < \varepsilon$$

for every  $n \geq N$ . ◀

**NB:** The  $N$  we pick depends on the  $\varepsilon$  we start with. Although many choices of  $N$  will work, we have to make sure it works for our particular  $\varepsilon$ .

**12.5 The Sequence**  $\{1 + 1/\sqrt{n}\}$ 

► **Example 12.5.1: The Sequence  $\{1 + 1/\sqrt{n}\}$  Converges.** This sequence has limit 1. Set  $x_n = 1 + 1/\sqrt{n}$  and consider  $|x_n - 1| = 1/\sqrt{n}$ . Choose any  $\varepsilon > 0$ .

We want  $1/\sqrt{n} < \varepsilon$ , meaning  $1/\varepsilon < \sqrt{n}$ , or equivalently  $1/\varepsilon^2 < n$ . So we pick  $N > 1/\varepsilon^2$ .

Then for  $n \geq N > 1/\varepsilon^2$ ,

$$|x_n - 1| = \sqrt{\frac{1}{n}} \leq \sqrt{\frac{1}{N}} < \sqrt{\frac{1}{\varepsilon^{-2}}} = \varepsilon.$$

It follows that for  $n \geq N$ ,

$$|x_n - 1| = \frac{1}{\sqrt{n}} < \varepsilon,$$

which tells us that  $\lim_n x_n = 1$ . ◀

## 12.6 Some Sequences Don't Converge

But do all sequences converge? What if we get one that doesn't? What if we try to show it converges to something?

► **Example 12.6.1: A Sequence without a Limit.** The sequence  $x_n = (-1)^n$  does not converge to anything. Examination of the sequence reveals the problem. It is  $\{-1, +1, -1, +1, -1, \dots\}$ . It never settles down but continues to bounce back-and-forth, back-and-forth, back-and-forth between  $-1$  and  $+1$ .

There must be an  $\varepsilon$  where the definition fails. The key thing is that  $\varepsilon$  must be small enough that only one of  $+1$  and  $-1$  can be near the same point. We could try to guess an  $\varepsilon$  where the definition fails.

Or, we could try to show it converges and see where things go wrong. So we try to make the sequence converge. Suppose the sequence has limit  $x$  and take  $\varepsilon > 0$ .

In that case, there is an  $N$  with  $|x_n - x| < \varepsilon$  for all  $n \geq N$ . Then also  $|x_{n+1} - x| < \varepsilon$ . Using the triangle inequality, we find

$$|x_n - x_{n+1}| \leq |x_n - x| + |x - x_{n+1}| < 2\varepsilon.$$

But  $|x_n - x_{n+1}| = 2$ , so  $2 < 2\varepsilon$ . It only works if  $\varepsilon > 1$ ! This fails for any  $\varepsilon \leq 1$ .

We've shown that  $x_n$  cannot converge to any limit because we cannot squeeze the terms closer to any limit than 1 unit. Any  $\varepsilon \leq 1$  will fail. The sequence  $\{(-1)^n\}$  is a sequence without a limit. ◀

This is not the only way convergence can fail. Sequences such as  $\{1, 2, 3, \dots\}$  ( $x_n = n$ ) and  $\{(-1)^n n\} = \{-1, 2, -3, 4, \dots\}$  also don't have limits.

## 12.7 Limits are Unique

What about the sequences that bounce around? Can a sequence have two limits? No. A sequence can only converge to one limit. The reasoning is similar to that in Example 12.6.1, where  $\{(-1)^n\}$  did not converge.

We will prove this for sequences in  $\mathbb{R}$ , but it is true in considerable generality, in metric spaces. The statement and proof of the general case are very similar to the following theorem.

**Theorem 12.7.1.** *Let  $\{x_n\}$  be a sequence of real numbers. Suppose  $x_n \rightarrow x$  and  $x_n \rightarrow y$ . Then  $x = y$ .*

**Proof.** **By way of contradiction, suppose  $x \neq y$ .**

Set  $\varepsilon = |x - y|/2$ . By the contradiction hypothesis,  $\varepsilon > 0$ . Since  $x_n \rightarrow x$ , there is an  $N_1$  with  $|x_n - x| < \varepsilon$  for  $n \geq N_1$ . Take  $N_2 \geq N_1$  with  $|x_n - y| < \varepsilon$  for  $n \geq N_2$ . Then for  $n \geq N_2 \geq N_1$ , both  $|x_n - x| < \varepsilon$  and  $|x_n - y| < \varepsilon$ . It follows that

$$|x - y| \leq |x - x_n| + |x_n - y| < \varepsilon + \varepsilon = |x - y|.$$

But this is impossible, **so  $x \neq y$  is impossible.**

The only possibility remaining is  $x = y$ . ■

## 12.8 Limits and Inequalities

One important property of limits in  $\mathbb{R}$  is that they preserve the weak order relations,  $\leq$  and  $\geq$ .

**Theorem 12.8.1.** *Let  $\{x_n\}$  and  $\{y_n\}$  be sequences of real numbers with  $x_n \leq y_n$  for every  $n$ . Then  $\lim_n x_n \leq \lim_n y_n$ .*

**Proof.** Let  $x = \lim_n x_n$  and  $y = \lim_n y_n$ . We will prove this by contradiction.

**Suppose that  $x > y$ .** Then we can set  $\varepsilon = (x - y)/2 > 0$ . Because  $x_n \rightarrow x$ , we can choose  $N_1$  with  $|x_n - x| < \varepsilon$  for  $n \geq N_1$ . Then we choose  $N_2 \geq N_1$  so that  $|y_n - y| < \varepsilon$  for  $n \geq N_2$ . We already know that  $|x_n - x| < \varepsilon$  for  $n \geq N_2 > N_1$ , so both inequalities hold for  $n \geq N_2$ .

Now for  $n \geq N_2$ ,  $|x - x_n| < \varepsilon$  and  $|y - y_n| < \varepsilon$ . These imply  $x < x_n + \varepsilon$  and  $y_n < y + \varepsilon$ . Then for  $n \geq N_2$ ,

$$\begin{aligned}
 x &< x_n + \varepsilon && \text{since } |x_n - x| < \varepsilon \\
 &\leq y_n + \varepsilon && \text{by hypothesis} \\
 &< y + 2\varepsilon && \text{since } y_n < y + \varepsilon \\
 &= y + (x - y) && \text{definition of } \varepsilon \\
 &= x.
 \end{aligned}$$

But  $x < x$  is impossible. It follows **that  $x > y$  cannot be true.**

Therefore  $x \leq y$ . ■

## 12.9 More Limit Inequalities

The result also holds if we reverse the inequalities.

**Theorem 12.9.1.** *Let  $\{x_n\}$  and  $\{y_n\}$  be sequences of real numbers with  $x_n \geq y_n$ . Then  $\lim_n x_n \geq \lim_n y_n$ .*

**Proof.** Adapt the proof of Theorem 12.8.1. ■

The corollary follows immediately.

**Corollary 12.9.2.** *Let  $\{x_n\}$  and  $\{y_n\}$  be convergent sequences of real numbers with  $x_n < y_n$  (or  $x_n > y_n$ ) for every  $n$ . Then  $\lim_n x_n \leq \lim_n y_n$  (or  $\lim_n x_n \geq \lim_n y_n$ ).*

Of course, Corollary 12.9.2 is weaker than Theorem 12.9.1. The only reason we stated it to emphasize that we can't strengthen them to obtain a strong inequality such as  $\lim_n x_n > \lim y_n$ . The following example demonstrates this.

► **Example 12.9.3: Inequality Counterexample.** Let

$$x_n = 1 + \frac{2}{n+1} \quad \text{and} \quad y_n = 1 + \frac{1}{n+1}.$$

Then  $x_n > y_n$ , but the limits are equal:  $\lim_n x_n = 1 = \lim_n y_n$ . ◀

## 12.10 Convergence in Metric Spaces

Before we look at too many results for  $\mathbb{R}$ , let's upgrade the definition of convergence. We will skip the special cases of Euclidean  $\mathbb{R}^m$  and even general normed spaces. We head directly to metric spaces, which are a natural home for convergent sequences.

Of course, every normed vector space is a metric space, so we are not really skipping them. Our new definition of convergence will work in normed spaces too. So how do we change the definition? In the old definition, we used  $|x_n - x|$ , which is the distance  $d(x_n, x)$ . Switching them is the only change we need!

**Convergent Sequences in Metric Spaces.** Let  $\{x_n\}$  be a sequence in a metric space  $(X, d)$ . We say that  $x_n$  *converges to*  $x$ , written  $x_n \rightarrow x$ , or  $\lim_{n \rightarrow \infty} x_n = x$ , if for every  $\varepsilon > 0$ , there is a positive integer  $N$  with  $d(x_n, x) < \varepsilon$  for every  $n \geq N$ . A sequence is *convergent* if it converges to some  $x \in X$ .

If we specialize to  $\mathbb{R}^m$ , the distance becomes  $\|x_n - x\|$ . Finally, in  $\mathbb{R}$ , the condition is exactly what we used earlier, that  $|x_n - x| < \varepsilon$  for all  $n \geq N$ .

### 12.11 Product Spaces

We will sometimes be interested in functions of two or more variables. We expand our notion of convergence to include such cases.

**Product Space.** Let  $X_1, X_2, \dots, X_k$  be sets. The *product space*

$$X_1 \times X_2 \times \cdots \times X_k$$

is the set of all  $k$ -tuples  $(x_1, \dots, x_k)$  with  $x_i \in X_i$  for all  $i = 1, \dots, k$ .

One product space you are familiar with is  $\mathbb{R}^m$ . We can write

$$\mathbb{R}^m = \overbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}^{m \text{ times}}.$$

Product spaces do not have to use the same set over and over. For example,

$$X = (0, 1) \times \mathbb{R}^2$$

is a product space.

## 12.12 Product Convergence

**Product Convergence.** Let  $X_1, X_2, \dots, X_k$  be metric spaces and consider their product space

$$X = X_1 \times X_2 \times \cdots \times X_k.$$

A sequence of  $k$ -tuples  $\{(\mathbf{x}_1^n, \dots, \mathbf{x}_k^n)\}_{n=1}^{\infty}$  *product converges* to the  $k$ -tuple  $(\mathbf{x}_1, \dots, \mathbf{x}_k)$  if  $\mathbf{x}_i^n \rightarrow \mathbf{x}_i$  for every  $i = 1, \dots, k$ .

We can use the shorthand  $\mathbf{x}^n = (\mathbf{x}_1^n, \dots, \mathbf{x}_k^n)$  and  $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$ . In that case, we write  $\mathbf{x}^n \xrightarrow{p} \mathbf{x}$  to denote product convergence.

► **Example 12.12.1:**  $\mathbb{R}^m$ . We previously treated  $\mathbb{R}^m$  as a normed vector space using any of the  $\ell_p$  norms. It can also be thought of as a product space,

$$\mathbb{R}^m = \overbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}^{m \text{ times}}$$

where the distance on each copy of  $\mathbb{R}$  is  $d(x, y) = |x - y|$ . Then  $\mathbf{x}_n \xrightarrow{p} \mathbf{x}$  means that for every  $i = 1, \dots, m$ ,  $x_i^n \rightarrow x_i$ . ◀

### 12.13 Convergence in $\mathbb{R}^m$

We now have two definitions of convergence in  $\ell_2^m = (\mathbb{R}^m, \|\cdot\|_2)$ , one using the  $\ell_2$  norm and the other treating  $\mathbb{R}^m$  as a product space.

In fact, both are the same. Convergence in  $\ell_2^m$  can be thought of as convergence in the  $\ell_2$  norm, or it can be thought of as convergence in every coordinate. We'll use the notation  $\mathbf{x}^n \xrightarrow{\ell_2} \mathbf{x}$  to indicate  $\ell_2$  convergence.

**Theorem 12.13.1.** *Let  $\{\mathbf{x}^n\}$  be a sequence in  $\ell_2^m$ . Then  $\mathbf{x}^n \xrightarrow{\ell_2} \mathbf{x}$  if and only if  $\mathbf{x}^n \xrightarrow{p} \mathbf{x}$ .*

**Proof. Only if case ( $\Rightarrow$ ):** Suppose  $\mathbf{x}^n \rightarrow \mathbf{x}$ . Let  $\varepsilon > 0$ . We can choose  $N$  with  $\|\mathbf{x}^n - \mathbf{x}\|_2 < \varepsilon$  for  $n \geq N$ . Then for  $n \geq N$ ,

$$\begin{aligned} |x_i^n - x_i| &= \left( |x_i^n - x_i|^2 \right)^{1/2} \\ &\leq \left( \sum_{j=1}^m |x_j^n - x_j|^2 \right)^{1/2} \\ &= \|\mathbf{x}^n - \mathbf{x}\|_2 \\ &< \varepsilon \end{aligned}$$

for all  $n \geq N$ . It follows that  $\lim_n x_i^n = x_i$  for each  $i = 1, \dots, m$ , showing that  $\mathbf{x}^n \xrightarrow{p} \mathbf{x}$ .

Proof continues ...

### 12.14 Proof of Theorem 12.13.1, If Case

If case ( $\Leftarrow$ ). Suppose that  $\mathbf{x}^n \xrightarrow{p} \mathbf{x}$ . This means that for each  $i = 1, \dots, m$ ,  $\lim_n x_i^n = x_i$ . Choose any  $\varepsilon > 0$ . For each  $i$ , we can find an  $N_i$  with

$$|x_i^n - x_i| < \frac{\varepsilon}{m^{1/2}} \quad (12.14.1)$$

whenever  $n \geq N_i$ . Let  $N = \max_i N_i$ . Then for  $n \geq N \geq N_i$ , equation (12.14.1) holds for every  $i = 1, \dots, m$ . So for all  $n \geq N$ ,

$$\begin{aligned} \|\mathbf{x}^n - \mathbf{x}\|_2 &= \left( \sum_{j=1}^m |x_j^n - x_j|^2 \right)^{1/2} \\ &< \left( \sum_{j=1}^m \left( \frac{\varepsilon}{m^{1/2}} \right)^2 \right)^{1/2} \\ &= \left( m \left( \frac{\varepsilon^2}{m} \right) \right)^{1/2} \\ &= \varepsilon \end{aligned}$$

showing that  $\mathbf{x}^n \xrightarrow{\ell_2} \mathbf{x}$ . ■

### 12.15 Open Balls

As with sequences in  $\mathbb{R}$ , when  $\mathbf{x}_n \rightarrow \mathbf{x}$  in a metric space  $(X, d)$ , the sequence eventually stays near the point  $\mathbf{x}$ . We'll now use a slightly different method to describe this that focuses a little more on where the sequence is.

**Open Balls.** Let  $(X, d)$  be a metric space. For  $\mathbf{x} \in X$  and  $\varepsilon > 0$ , we define the *open ball of radius  $\varepsilon$  about  $\mathbf{x}$* ,  $B_\varepsilon(\mathbf{x})$  by

$$B_\varepsilon(\mathbf{x}) = \{\mathbf{y} \in X : d(\mathbf{x}, \mathbf{y}) < \varepsilon\}.$$

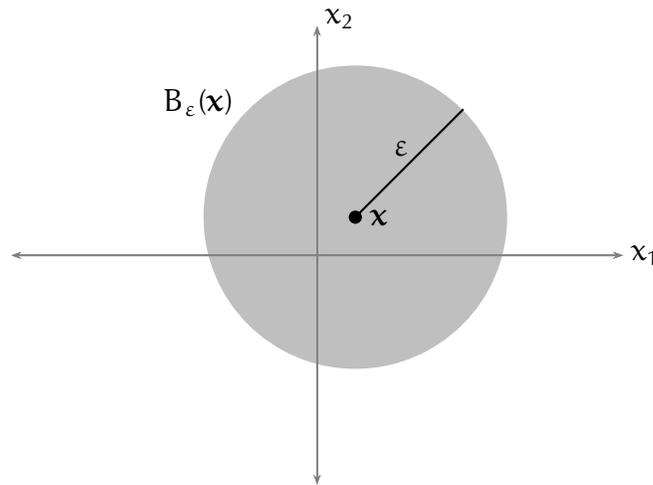
The set of points with  $d(\mathbf{x}, \mathbf{y}) = \varepsilon$  is **not** included in  $B_\varepsilon(\mathbf{x})$ . These points are included in the *closed ball*

$$\overline{B_\varepsilon(\mathbf{x})} = \{\mathbf{y} \in X : d(\mathbf{x}, \mathbf{y}) \leq \varepsilon\}.$$

We make a distinction between the ball and its boundary, the *sphere*,  $\{\mathbf{y} \in X : d(\mathbf{x}, \mathbf{y}) = \varepsilon\}$ . In  $\mathbb{R}^m$  we can refer to the  $m$ -ball and  $(m - 1)$ -sphere. Here the dimension  $m$  refers to dimension of the set (which we have not covered) rather than the dimension of the space. In two dimensions, the ball is sometimes called a *disk* and its boundary a circle.

### 12.16 An Open Ball in $\mathbb{R}^2$

The  $\mathbb{R}^2$  case, where  $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2$ , is illustrated in Figure 12.16.1.



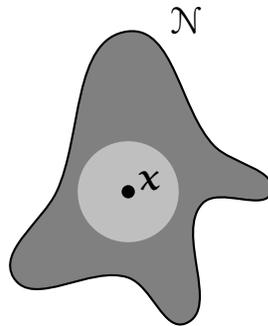
**Figure 12.16.1:** The open  $\varepsilon$ -ball about  $\mathbf{x}$  in  $\mathbb{R}^2$  is the disk of radius  $\varepsilon$  centered at  $\mathbf{x}$ . The circle of points at distance  $\varepsilon$  from  $\mathbf{x}$  are not included in the open disk, but are part of the closed disk.

### 12.17 Neighborhoods

A set  $\mathcal{N} \subset X$  is a *neighborhood* or *nhood* of  $\mathbf{x} \in X$  if there is  $\varepsilon > 0$  with  $B_\varepsilon(\mathbf{x}) \subset \mathcal{N}$ . In other words, a neighborhood of a point contains an open ball about that point (and all smaller open balls about it).

It is easy to see that  $B_\varepsilon(\mathbf{x})$  is not only a neighborhood of  $\mathbf{x}$ , but also of any  $\mathbf{y} \in B_\varepsilon(\mathbf{x})$ . This follows because the ball of radius  $\delta = \varepsilon - \|\mathbf{y} - \mathbf{x}\|$  is contained in  $B_\varepsilon(\mathbf{x})$ . [Hint: Use the triangle inequality.]

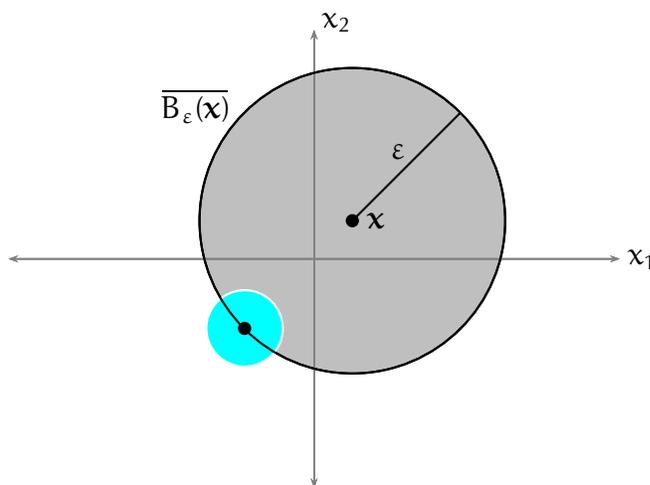
#### A Neighborhood of $\mathbf{x}$



**Figure 12.17.1:** The set  $\mathcal{N}$  is a neighborhood of  $\mathbf{x}$  because it contains an open ball about  $\mathbf{x}$  (shown in light gray).

### 12.18 Closed Balls as Neighborhoods

A neighborhood of  $\mathbf{x}$  only has to contain a ball around  $\mathbf{x}$ , not about any other point. In particular, the **closed**  $\varepsilon$ -ball in  $\mathbb{R}^m$ ,  $\overline{B_\varepsilon(\mathbf{x})}$  is a neighborhood of  $\mathbf{x}$ . It is also a neighborhood of every point  $\mathbf{y}$  with  $d(\mathbf{x}, \mathbf{y}) < \varepsilon$ , but not of points with  $d(\mathbf{x}, \mathbf{y}) = \varepsilon$ . This is illustrated in Figure 12.18.1.



**Figure 12.18.1:** The closed  $\varepsilon$ -ball about  $\mathbf{x}$  in  $\mathbb{R}^2$  is the closed disk of radius  $\varepsilon$  centered at  $\mathbf{x}$ . The circle of points at distance  $\varepsilon$  from  $\mathbf{x}$  are included in the closed disk. Although the closed disk is a neighborhood of each point in the open disk, it is not a neighborhood of any point on the boundary. As the figure shows, open balls about such boundary points stick out of the closed disk.

To see analytically that the closed  $\varepsilon$ -ball is not a neighborhood of any boundary point, take any  $\mathbf{y}$  with  $d(\mathbf{x}, \mathbf{y}) = \varepsilon$  and consider points of the form  $\mathbf{y}_\delta = \delta(\mathbf{y} - \mathbf{x}) + \mathbf{y}$  for  $\delta > 0$ . Now  $\|\mathbf{y}_\delta - \mathbf{x}\| = (\delta + \varepsilon)$ , so  $\mathbf{y}_\delta \notin B_\varepsilon(\mathbf{y})$ . This shows that any open ball about  $\mathbf{y}$  must contain points outside of  $B_\varepsilon(\mathbf{x})$ .

### 12.19 Convergence in Metric Spaces

We can now restate the definition of a convergent sequence that applies in any metric space  $(X, d)$ .

**Convergent Sequences in Metric Spaces II.** Let  $\{\mathbf{x}_n\}_{n=1}^{\infty}$  be a sequence in a metric space  $(X, d)$ . The sequence  $\{\mathbf{x}_n\}_{n=1}^{\infty}$  *converges to*  $\mathbf{x}$  if for every  $\varepsilon > 0$ , there is a positive integer  $N$  with  $\mathbf{x}_n \in B_\varepsilon(\mathbf{x})$  for every  $n \geq N$ .

Alternatively,  $\{\mathbf{x}_n\}$  *converges to*  $\mathbf{x}$  if for every neighborhood  $\mathcal{N}$  of  $\mathbf{x}$ , there is a positive integer  $N$  with  $\mathbf{x}_n \in \mathcal{N}$  for every  $n \geq N$ .

Of course, saying that  $\mathbf{x}_n \in B_\varepsilon(\mathbf{x})$  for all  $n \geq N$  is exactly the same as saying  $d(\mathbf{x}_n, \mathbf{x}) < \varepsilon$  for all  $n \geq N$ . The point is that the new version makes you think about convergence a little differently, putting the focus on the  $\varepsilon$ -balls  $B_\varepsilon(\mathbf{x})$  or even the neighborhoods of  $\mathbf{x}$ , and de-emphasizing the metric itself. It's a more topological way of thinking about convergence.

## 12.20 Accumulation or Cluster Points

There are several ways that a sequence can fail to converge. It could head off to infinity, oscillate, or continually leave and return to the neighborhood of some point. The last two are our focus of interest.

**Accumulation Point, Cluster Point.** We say that  $\mathbf{x}$  is an *accumulation point* or *cluster point* of  $\{\mathbf{x}_n\}_{n=1}^{\infty}$  if for every  $\varepsilon > 0$  and every positive integer  $N$ , there is  $n > N$  with  $\mathbf{x}_n \in B_\varepsilon(\mathbf{x})$ .

Alternatively,  $\mathbf{x}$  is a cluster point if for every neighborhood  $\mathcal{N}$  of  $\mathbf{x}$  and every positive integer  $N$ , there is  $n > N$  with  $\mathbf{x}_n \in \mathcal{N}$ .

A point  $\mathbf{x}$  is an accumulation point of a sequence if the sequence continually returns to any  $\varepsilon$ -ball about  $\mathbf{x}$ . It is not required to stay there, but does have to return there.

Both  $+1$  and  $-1$  are accumulation points of the sequence  $x_n = (-1)^n$ , even though the sequence has no limit. The accumulation point doesn't have to be part of the sequence. Consider the sequence defined by

$$x_n = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 1 + 1/n & \text{if } n \text{ is even} \end{cases}$$

The points  $0$  and  $1$  are accumulation points even though they are not part of the sequence.

Can we get a sequence that converges to an accumulation point? Yes! If  $\mathbf{x}$  is an accumulation point of  $\{\mathbf{x}_n\}$ , we can find an  $\mathbf{x}_{n_1}$  with  $d(\mathbf{x}_{n_1}, \mathbf{x}) < 1$ . Then we can find  $n_2 > n_1$  with  $d(\mathbf{x}_{n_2}, \mathbf{x}) < 1/2$ . We continue this with  $n_k > n_{k-1}$  and  $d(\mathbf{x}_{n_k}, \mathbf{x}) < 1/k$  for every  $k$ . If we then set  $\mathbf{y}_k = \mathbf{x}_{n_k}$ ,  $\mathbf{y}_k$  is a sequence that converges to  $\mathbf{x}$ .

## 12.21 Subsequences

So what have we done? We have constructed a new sequence by plucking out certain elements of the original sequence, and we have done this in a way that guarantees that as we go out the new sequence, we are also moving out in the original sequence. This is an example of a *subsequence*.

**Subsequence.** A sequence  $\{\mathbf{y}_k\}_{k=1}^{\infty}$  is a *subsequence* of  $\{\mathbf{x}_n\}_{n=1}^N$  if there is an increasing set of positive integers,

$$n_1 < n_2 < n_3 < \cdots$$

with  $\mathbf{y}_k = \mathbf{x}_{n_k}$  for every  $k = 1, 2, 3, \dots$ .

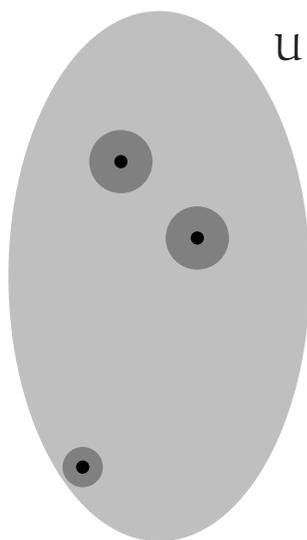
We often don't bother writing down  $\mathbf{y}_k$ , but instead just use  $\mathbf{x}_{n_k}$  to indicate the subsequence. For example, if  $x_n = (-1)^n$  defines a sequence, it is not convergent but has two cluster points,  $-1$  and  $+1$ . The subsequence  $x_{2k+1}$  (i.e.,  $n_k = 2k + 1$ ) converges to  $-1$  and  $x_{2k}$  converges to  $+1$ . These are not the only convergent subsequences. The subsequence  $x_{2k^2+1}$  also converges to  $-1$ .

Finally, an alternative way to characterize cluster points is that a point  $\mathbf{x}$  is a cluster (accumulation) point of  $\{\mathbf{x}_n\}$  if and only if there is a subsequence of  $\{\mathbf{x}_n\}$  with limit  $\mathbf{x}$ .

**12.22 Open Sets****10/4/22**

**New Homework:** Problems 12.15, 12.16, 12.20, 12.21, and 13.17 are due on **Tuesday, October 11**.

A set  $U$  is *open* if it contains a ball about each point in the set. Equivalently,  $U$  is open if it contains a neighborhood of each of its points, or more directly,  $U$  is open if it **is** a neighborhood of each of its points.



**Figure 12.22.1:** The ellipse  $U$  is an open set. Three points are illustrated, along with darker balls that both contain the points and are themselves contained in the ellipse.

The open interval  $(a, b) \subset \mathbb{R}$  is open. If  $a < x < b$ , let  $\varepsilon = \min\{b - x, x - a\}$ . Then  $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon) \subset (a, b)$ . The choice of  $\varepsilon$  ensures that  $B_\varepsilon(x) \subset (a, b)$ .

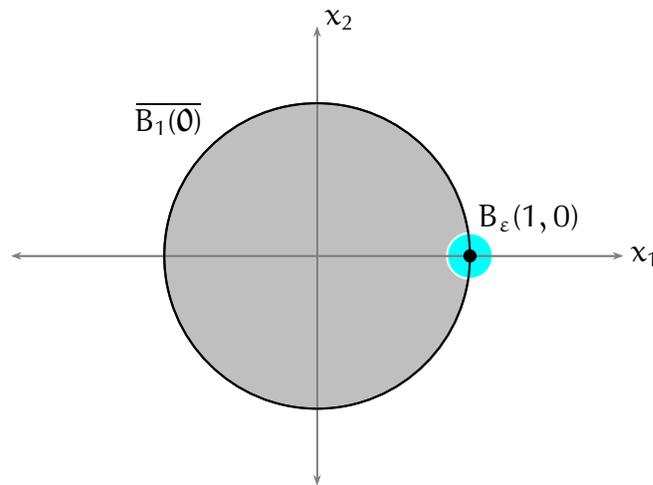
Like open balls, open sets absorb the tails of convergent sequences.

**Theorem 12.22.2.** Let  $U$  be an open set and  $x \in U$ . If  $x_n \rightarrow x$ , then there is an  $N$  with  $x_n \in U$  for  $n \geq N$ .

**Proof.** Since  $U$  is open and  $x \in U$ , there is an  $\varepsilon > 0$  with  $x \in B_\varepsilon(x) \subset U$ . Because  $x_n \rightarrow x$ , there is an  $N$  with  $x_n \in B_\varepsilon(x)$  for  $n \geq N$ . But  $B_\varepsilon(x) \subset U$ , so  $x_n \in U$  for  $n \geq N$ . ■

### 12.23 Closed Balls are Not Open

The closed ball of radius 1,  $\overline{B_1(\mathbf{0})} = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| \leq 1\}$  is not open. The problem is that there is no room to fit in a ball about any point on the boundary. For example, if  $\mathbf{x} = (1, 0)$ , any  $B_\varepsilon(1, 0)$  will contain the point  $(1 + \varepsilon/2, 0) \notin \overline{B_1(\mathbf{0})}$ .



**Figure 12.23.1:** Notice how the cyan  $\varepsilon$ -ball about  $(1, 0)$  protrudes out of  $\overline{B_1(\mathbf{0})}$ , showing that  $\overline{B_1(\mathbf{0})}$  is not open.

**12.24 Sets that are Not Open**

There are many other sets that are not open. Here are a few.

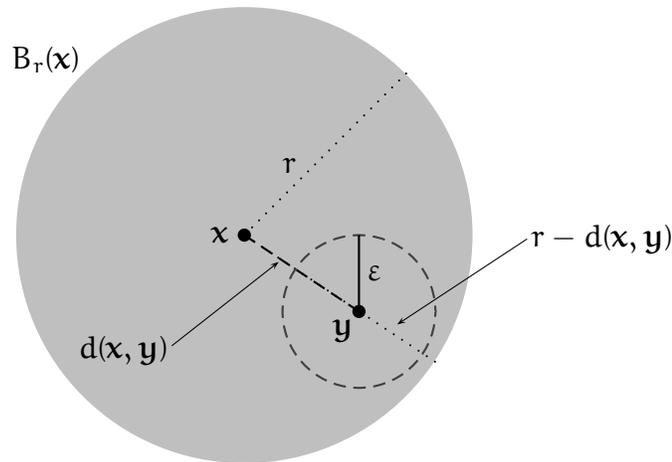
- A singleton,  $S = \{\mathbf{x}\}$  in  $\mathbb{R}^m$  is not open in the usual topology on  $\mathbb{R}^m$  (using the Euclidean norm) as any  $\varepsilon$ -ball about  $\mathbf{x}$  will also contain points not in  $S$ .
- The half-open interval  $(0, 1]$  is not open. The problem here is that any open ball containing 1 also contains points greater than one, and those are not in the set. That means that no open ball about one is contained in  $(0, 1]$ .
- The rectangle  $[-1, 1] \times (-2, 2)$  is not open. Once again, the boundary is a problem. There are no open balls about  $(-1, 0)$  that are contained in the rectangle.

## 12.25 Open Balls are Open!

Open balls are open sets.

**Theorem 12.25.1.** Any open ball  $B_r(\mathbf{x})$ ,  $r > 0$ , is an open set.

Before proving this, it is helpful to draw a diagram.



**Figure 12.25.2:** To show  $B_r(\mathbf{x})$  is open, we need to show that for any point  $\mathbf{y} \in B_r(\mathbf{x})$  we can find an  $\varepsilon$ -ball about  $\mathbf{y}$  that fits inside  $B_r(\mathbf{x})$ , as illustrated here. Since  $r$  is the radius of  $B_r(\mathbf{x})$  and  $d(\mathbf{y}, \mathbf{x})$  is the distance between  $\mathbf{x}$  and  $\mathbf{y}$  (dashed), the distance from  $\mathbf{y}$  to the boundary (dotted) is  $r - d(\mathbf{y}, \mathbf{x})$ .

**Proof.** Let  $\mathbf{y} \in B_r(\mathbf{x})$ . Based on Figure 12.25.2, we should be to show that  $B_\varepsilon(\mathbf{y}) \subset B_r(\mathbf{x})$  for any  $\varepsilon < r - d(\mathbf{y}, \mathbf{x})$ . So take  $\varepsilon$  with  $0 < \varepsilon < r - d(\mathbf{y}, \mathbf{x})$  and let  $\mathbf{z} \in B_\varepsilon(\mathbf{y})$ . We must show  $\mathbf{z} \in B_r(\mathbf{x})$ .

Now

$$\begin{aligned}
 d(\mathbf{z}, \mathbf{x}) &\leq d(\mathbf{z}, \mathbf{y}) + d(\mathbf{y}, \mathbf{x}) && \text{triangle inequality} \\
 &< \varepsilon + d(\mathbf{y}, \mathbf{x}) && \mathbf{z} \in B_\varepsilon(\mathbf{y}) \\
 &< (r - d(\mathbf{y}, \mathbf{x})) + d(\mathbf{y}, \mathbf{x}) && \text{choice of } \varepsilon \\
 &= r.
 \end{aligned}$$

It follows that any  $\mathbf{z} \in B_\varepsilon(\mathbf{y})$  is also in  $B_r(\mathbf{x})$ , establishing that  $B_\varepsilon(\mathbf{y}) \subset B_r(\mathbf{x})$ , completing the proof. ■

## 12.26 The Topology of Open Sets

There are three fundamental properties that open sets have. These properties are so important, that in more general works they are part of the definition of open sets.

**Topology.** Let  $\mathcal{T}$  be a collection of subsets of  $X$ . We say that  $\mathcal{T}$  is a *topology on  $X$*  if:

1. Any union of sets in  $\mathcal{T}$  is in  $\mathcal{T}$ .
2. Any finite intersection of sets in  $\mathcal{T}$  is in  $\mathcal{T}$ .
3. The empty set and  $X$  are in  $\mathcal{T}$ .

The sets in a topology  $\mathcal{T}$  are referred to as the *open sets* of topology  $\mathcal{T}$ .

### 12.27 Smallest and Largest Topologies

The simplest possible topology on  $X$  is  $\mathcal{T} = \{\emptyset, X\}$ . Any other topology must contain this one. It is called the *trivial topology*, the *indiscrete topology*, or the *coarsest* or *weakest possible topology*. It is the smallest possible topology.

The largest possible topology on  $X$  is to let  $\mathcal{T} = \{\text{all subsets of } X\}$ . Any other topology is a subset of this one. It is called the *discrete topology*, or the *finest* or *strongest possible topology*.

The discrete topology is a metric topology using the discrete metric

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0 & \text{when } \mathbf{x} = \mathbf{y} \\ 1 & \text{otherwise} \end{cases}$$

The open balls in the discrete topology are a bit surprising. They consist of a single point whenever the radius is less than one. Then  $B_{1/2}(\mathbf{x}) = \{\mathbf{x}\}$ . It follows that every set  $S$  is open because if  $\mathbf{x} \in S$ ,  $B_{1/2}(\mathbf{x}) \subset S$ .

## **12.28 Basic Properties of Open Sets**

One important result is that the open sets as defined by  $\varepsilon$ -balls form a topology.

**Theorem 12.28.1.** *Let  $(X, d)$  be a metric space. The open sets defined using  $\varepsilon$ -balls form a topology for  $(X, d)$ . That is:*

1. *Any union of open sets is an open set.*
2. *Any finite intersection of open sets is an open set.*
3. *The empty set and  $X$  are open sets.*

**12.29 Proof of Theorem 12.28.1**

**Proof.** (1) **Any union of open sets is an open set.** To prove this, let  $\mathcal{U}$  be any collection of open sets in  $X$  and

$$V = \bigcup_{U \in \mathcal{U}} U$$

be the union of all sets in  $\mathcal{U}$ . If  $\mathbf{x} \in V$ , we can find  $U_{\mathbf{x}} \in \mathcal{U}$  with  $\mathbf{x} \in U_{\mathbf{x}}$ . Since  $U_{\mathbf{x}}$  is open, there is  $\varepsilon > 0$  with  $B_{\varepsilon}(\mathbf{x}) \subset U_{\mathbf{x}} \subset V$ . This shows that  $V$  is open.

(2) **Any finite intersection of open sets is an open set.** To prove this, consider a finite collection  $U_1, \dots, U_k$  of open sets in  $\mathcal{U}$  and set

$$V = \bigcap_{i=1}^k U_i.$$

If  $\mathbf{x} \in V$ , then for every  $i = 1, \dots, k$ ,  $\mathbf{x} \in U_i$ . As each  $U_i$  is open, there is  $\varepsilon_i > 0$  with  $B_{\varepsilon_i}(\mathbf{x}) \subset U_i$ . Let  $\varepsilon = \min_i \varepsilon_i > 0$ . Then  $B_{\varepsilon}(\mathbf{x}) \subset B_{\varepsilon_i}(\mathbf{x}) \subset U_i$  for each  $i = 1, \dots, k$ , showing that  $B_{\varepsilon}(\mathbf{x}) \subset \bigcap_{i=1}^k U_i = V$ . This shows that  $V$  is open.

(3) **The empty set and  $X$  are open sets.** It is vacuously true that  $\emptyset$  is open. What does that mean? It means you can't show  $\emptyset$  is not open because that would require it have something in it, which it doesn't. Suppose  $\emptyset$  was not open. Then there would be  $\mathbf{x} \in \emptyset$  with  $B_{\varepsilon}(\mathbf{x}) \not\subset \emptyset$  for all  $\varepsilon > 0$ . However, because  $\emptyset$  is empty, there can't be any such  $\mathbf{x} \in \emptyset$ .

As for  $X$ , let  $\mathbf{x} \in X$ . Since  $X$  contains all balls around  $\mathbf{x}$  for every  $\mathbf{x} \in X$ ,  $X$  is open. ■

**12.30 The Interior of a Set**

Every set has an open set associated with it, the interior of the set. However, the interior may be empty.

**Interior.** Let  $S$  be a set. The *interior of  $S$* , written  $S^0$  or  $\text{int } S$ , is the union of all open sets contained in  $S$ .

As the union of open sets,  $S^0$  is open. In fact, it is the largest open set contained in  $S$ .

**Theorem 12.30.1.** *Let  $S$  be a set. Then the interior  $S^0$  is the largest open set contained in  $S$ .*

**Proof.** Let  $S_1$  be the largest open set contained in  $S$ . Since  $S^0$  is the union of all open sets contained in  $S$ , including  $S_1$ ,  $S^0 \supset S_1$ .

But  $S_1$  is the largest open set contained in  $S$ , so if  $U \subset S$  and  $U$  is open,  $U \subset S_1$ . Otherwise,  $U \cup S_1$  would be a larger open set contained in  $S$ . Because  $S_1$  contains all open sets in  $S$ , it contains their union  $S^0$ .

Now both  $S^0 \subset S_1$  and  $S_1 \subset S^0$ , so  $S_1 = S^0$ . ■

**12.31 More on Interiors**

When a set  $S$  is open,  $S^0 = S$ .

For sets in  $\mathbb{R}$ ,  $(a, b]^0 = [a, b]^0 = [a, b)^0 = (a, b)$ .

The interior in  $\mathbb{R}$  of the rational numbers  $\mathbb{Q}$  is empty. To see this note that if  $x \in \mathbb{Q}$ ,  $B_\varepsilon(x)$  will always contain irrational numbers of the form  $x + 1/n\sqrt{2}$  for  $n$  large enough whenever  $\varepsilon > 0$ . This means that  $\mathbb{Q}$  has no open subsets, and so the interior of  $\mathbb{Q}$ , the union of open sets contained in  $\mathbb{Q}$ , is empty.

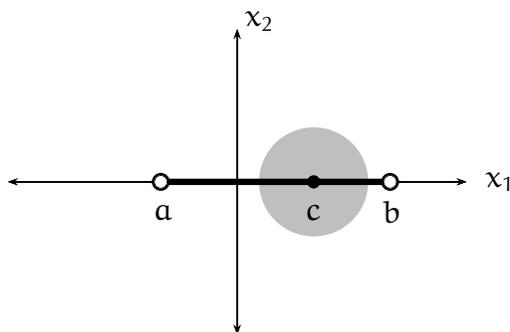
Informally, we can think of the interior of  $S$  as  $S$  with the boundary removed. Be careful, your intuition can fool you in arbitrary  $\mathbb{R}^m$ .

### 12.32 Interiors and the Ambient Space

It's not a good practice to think about interiors, or even whether a set is open, as being an intrinsic property of the set. The topological properties of a set can change depending on what topological space it is part of.

For example, consider an open interval  $(a, b) \subset \mathbb{R}$ . In  $\mathbb{R}$ ,  $(a, b) = B_r((a+b)/2)$  where  $r = (b-a)/2$ . As an open ball it is an open set. Intuitively, an open set is a set that does not include its boundary points. Here the boundary points are  $a$  and  $b$ , and are not part of the set.

Let's take that same set and embed it in  $\mathbb{R}^2$ . One way to do it is to define  $S = \{(x, y) : a < x < b, y = 0\}$ . But this set is not open at all. In fact,  $S^0 = \emptyset$ . See the Figure 12.32.1.



**Figure 12.32.1:** Here  $(a, b)$  has been embedded in  $\mathbb{R}^2$  as  $\{(x, y) : a < x < b, y = 0\}$ . Any point such as  $(c, 0)$  is not in the interior as any ball around it will include points above and below the embedded interval.

### 12.33 Closed Sets

Open sets are not the only way to characterize a topology. Closed sets can also do the job.

**Closed Set.** A set  $S$  is closed if whenever  $\{x_n\}$  is a convergent sequence in  $S$ , its limit  $\lim_n x_n$  is also in  $S$ .

Any singleton is a closed set.

**Theorem 12.33.1.** *A singleton  $S = \{x\}$  is a closed set.*

**Proof.** If  $\{x_n\}$  is a sequence in  $S$ , then each  $x_n = x$  must be  $x$ , so  $\lim_n x_n = x \in S$ . ■

### 12.34 Complements of Closed Sets

Suppose a set  $S$  is contained in the space  $X$ . The *complement* of  $S$  is  $S^c = \{\mathbf{x} \in X : \mathbf{x} \notin S\}$ .

**Theorem 12.34.1.** *Let  $(X, d)$  be a metric space. A set  $S \subset X$  is closed if and only if  $T = S^c$  is open*

**Proof.** Only if case ( $\Rightarrow$ ): Suppose  $S$  is closed. We must show that  $T = S^c$  is open. That is, for all  $\mathbf{x} \in T$ , there is  $\varepsilon > 0$  with  $B_\varepsilon(\mathbf{x}) \subset T$ . Take  $\mathbf{x} \in T$ .

**If not**, then for every  $\varepsilon > 0$ ,  $B_\varepsilon(\mathbf{x}) \not\subset T$ . That means that for every  $n$ , we can find  $\mathbf{x}_n \in B_{1/n}(\mathbf{x})$  with  $\mathbf{x}_n \notin T$ . In other words, there are  $\mathbf{x}_n \rightarrow \mathbf{x}$  with  $\mathbf{x}_n \in S$ . Since  $S$  is closed,  $\mathbf{x} \in S$ , implying  $\mathbf{x} \notin S^c = T$ . **This contradiction shows that  $T$  must be open.**

**If case ( $\Leftarrow$ ):** Suppose  $T$  is open and  $S = T^c$ . Let  $\mathbf{x}_n \in S$  with  $\mathbf{x}_n \rightarrow \mathbf{x}$ . We must show that  $\mathbf{x} \in S$ . **If not**,  $\mathbf{x} \in S^c = T$ . Since  $T$  is open, there is  $\varepsilon > 0$  with  $B_\varepsilon(\mathbf{x}) \subset T$ . Because  $\mathbf{x}_n \rightarrow \mathbf{x}$ , there is  $N > 0$  with  $\mathbf{x}_n \in B_\varepsilon(\mathbf{x}) \subset T = S^c$  for  $n \geq N$ . **This contradicts** the fact that  $\mathbf{x}_n \in S$  for all  $n$ . Therefore  $\mathbf{x} \in S$ , showing that  $S$  is closed. ■

### 12.35 Open? Closed? Both? Neither?

Theorem 12.34.1 does **not** say that every set is either open or closed.

The whole space and the empty set are **both** open and closed at the same time. They are the only such sets in any  $\mathbb{R}^m$ , and are commonly the only such sets.

Many sets are **neither** open **nor** closed. Let  $a, b \in \mathbb{R}$ ,  $a < b$ . In  $\mathbb{R}$ , the interval  $(a, b)$  is open,  $[a, b]$  is closed, and both of the half-open, half-closed intervals  $(a, b]$  and  $[a, b)$  are **neither open nor closed**.

We can find such sets in other spaces. In  $\mathbb{R}^2$ , consider the punctured closed disk  $S = \{\mathbf{x} \in \mathbb{R}^2 : 0 < \|\mathbf{x}\| \leq 1\}$ . This set is neither open nor closed. It is not closed because  $\mathbf{x}_n = (1/n, 1/n)$  is a sequence of points in  $S$  whose limit,  $(0, 0)$ , is not in  $S$ . It is not open because points such as  $(1, 0) \in S$ , and any  $\varepsilon$ -ball around  $(1, 0)$  will contain points not in  $S$  (e.g.,  $(0, 1 + \varepsilon/2)$ ).

### 12.36 Properties of Closed Sets

Just as open sets obey certain properties, so do closed sets.

**Theorem 12.36.1.** *Let  $(X, d)$  be a metric space. The closed sets obey:*

1. *Any finite union of closed sets is a closed set.*
2. *Any intersection of closed sets is a closed set.*
3. *The empty set and  $X$  are closed sets.*

**Proof.** Take complements and use Theorems 12.34.1 (complements of open sets) and 12.28.1 (properties of open sets). The result follows immediately. ■

Since singletons are closed sets, any finite union of singletons is closed too, meaning that all sets of the form  $\{x_1, \dots, x_n\}$  are closed.

The situation with countably and uncountably infinite sets of points is more complicated. Any convergent sequence, together with its limit, forms a countable closed sets. Without the limit, it is not closed.

There are other kinds of countable closed sets. An infinite set of isolated points in  $\mathbb{R}^2$  that is closed is the set of grid points in  $\mathbb{R}^2$ ,

$$\{(x, y) \in \mathbb{R}^2 : \text{both } x \text{ and } y \text{ are integers}\}.$$

This is closed because if  $(x_n, y_n)$  are grid points converging to  $(x, y)$ , then if  $\|(x_n, y_n) - (x, y)\| < 1$ ,  $x_n = x$  and  $y_n = y$ . It follows that any convergent sequence of grid points must obey  $(x_n, y_n) = (x, y)$  for  $n$  large enough, so that  $(x, y)$  is also a grid point.

### 12.37 Closures

Every set has a closed set associated with it, its closure.

**Closure.** Let  $S$  be a set. The *closure of  $S$* , written  $\bar{S}$  or  $\text{cl } S$ , is the smallest closed set containing  $S$ . Equivalently, it is the intersection of all closed sets containing  $S$ .

As with open sets, we could prove the equivalence of the two definitions of closure, but the arguments are pretty similar, so we won't. I do suggest you give it a try.

The closure always includes  $S$  itself.

**Lemma 12.37.1.** *If  $x \in S$ , then  $x \in \bar{S}$ .*

**Proof.** Consider the sequence  $x_n = x$ . This obviously converges to  $x$ , so  $x \in \bar{S}$ . ■

That is not the full story ...

### 12.38 Limit Points and Closure

Besides  $S$  itself,  $\bar{S}$  includes points that arise as limits of points in  $S$ .

**Limit Point.** Let  $S$  be a set. A point  $\mathbf{x}$  is a *limit point* of  $S$  if for all  $\varepsilon > 0$ ,  $B_\varepsilon(\mathbf{x})$  contains some point of  $S$  other than  $\mathbf{x}$ .

Equivalently, a point  $\mathbf{x}$  is a limit point of  $S$  if there is a sequence of points in  $S$  other than  $\mathbf{x}$  that converge to  $\mathbf{x}$ .

The closure of  $S$  consists of  $S$  together with its limit points.

**Theorem 12.38.1.** *The closure of  $S$  is the union of  $S$  and the set of limit points of  $S$ .*

**Proof.** Suppose  $T \supset S$  is closed. By definition of closed, every limit point of  $S$  is in  $T$ . It follows that the closure of  $S$  contains all limit points of  $S$  as well as  $S$  itself.

Suppose that  $\mathbf{x} \notin S$  is not a limit point of  $S$ . Then there is an  $\varepsilon > 0$  so that  $B_\varepsilon(\mathbf{x})$  contains no points of  $S$ . Then its complement,  $B_\varepsilon(\mathbf{x})^c$ , is a closed set containing  $S$  but not containing  $\mathbf{x}$ . Since  $\bar{S}$  is the intersection of all closed sets containing  $S$ ,  $\mathbf{x} \notin \bar{S}$ . Thus  $\bar{S}$  contains nothing other than  $S$  and its limit points. ■

**12.39 The Closure of the Open Ball**

In  $\mathbb{R}^m$ , or any normed vector space, the closure of any open ball is the corresponding closed ball.

► **Example 12.39.1: Closure of the Open Ball.** Let's find the closure of the open unit ball in a normed vector space  $(V, \|\cdot\|)$ . Set  $S = B_1(\mathbf{y}) = \{\mathbf{x} \in V : \|\mathbf{x} - \mathbf{y}\| < 1\}$ . First, if  $\mathbf{x}_n \in S$  converges to  $\mathbf{x}$ ,  $\|\mathbf{x}_n - \mathbf{y}\| < 1$ . By Theorem 13.21.1,  $\|\mathbf{x} - \mathbf{y}\| \leq 1$ . The only question is whether all of the points with  $\|\mathbf{x} - \mathbf{y}\| = 1$  are in the closure of  $S$ .

For  $\mathbf{x}$  with  $\|\mathbf{x} - \mathbf{y}\| = 1$ , consider  $\mathbf{x}_n = \mathbf{y} + (1 - \frac{1}{n})(\mathbf{x} - \mathbf{y})$ . Then  $\|\mathbf{x}_n - \mathbf{y}\| = (1 - \frac{1}{n})\|\mathbf{x} - \mathbf{y}\| < 1$ , so  $\mathbf{x}_n \in S$ . Also  $\mathbf{x}_n \rightarrow \mathbf{x}$ , so  $\mathbf{x}$  is a limit point of  $S$ .

In general,

$$\overline{B_\varepsilon(\mathbf{x})} = \{\mathbf{y} \in X : \|\mathbf{x} - \mathbf{y}\| \leq \varepsilon\}$$

is a closed set. ◀

**12.40 Example: The Cantor Set I****SKIPPED**

One famous closed set is the Cantor set. Start with the closed interval  $C_0 = [0, 1]$  and remove its open middle third,  $(1/3, 2/3)$ . That leaves the closed set  $C_1 = [0, 1/3] \cup [2/3, 1]$ . Then remove the open middle thirds of each part of  $C_1$  to get another closed set,

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

At each step,  $C_n$  is closed because it is a finite union of closed intervals. Each time, we form the next  $C_{n+1}$  by removing the open middle third of each interval. The *Cantor set*  $\mathfrak{C}$  is the intersection of the  $C_n$ .

$$\mathfrak{C} = \bigcap_{n=1}^{\infty} C_n.$$

As the intersection of non-empty nested closed sets, the Cantor set is closed.

One way to understand the Cantor set a bit better is to write the  $x \in [0, 1]$  in base-3 (ternary) form. I.e., we interpret the expression

$$0.d_1d_2d_3\dots = \sum_{n=1}^{\infty} d_n/3^n \quad (\text{base } 3)$$

rather than

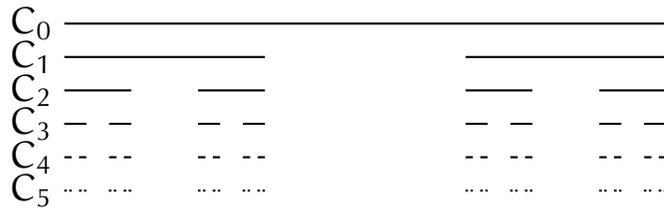
$$0.d_1d_2d_3\dots = \sum_{n=1}^{\infty} d_n/10^n \quad (\text{base } 10)$$

Notice that there may be two ways to write such numbers. We can write  $1/3 = 0.1000\dots = 0.02222\dots$

**12.41 Example: The Cantor Set II****SKIPPED**

Whenever we **must** write the ternary form of a number in  $[0, 1]$  in a way that uses a 1, we are in one of the middle thirds that is removed to form the Cantor set. Thus  $1/3 = 0.0222\dots$  is not removed by when  $(1/3, 2/3)$  is removed, but  $5/9 = 0.12$  is removed.

That means that the Cantor set consists of all  $x \in [0, 1]$  that can be written in ternary form using only 0 and 2. Since 1 is not involved, this expression is unique. Numbers starting with 0.0 are in the first third (including  $1/3 = 0.0222\dots$ ), while numbers starting 0.2 are in the last third (including  $2/3 = 0.2$ ), and so forth.



**Figure 12.41.1:** Here are the  $C_0, \dots, C_5$ , the first six sets defined by eliminating middle thirds, and whose intersection comprises the Cantor set  $\mathcal{C}$ .

## 12.42 Boundary of a Set

Another important concept is the boundary of a set.

**Boundary.** The *boundary of a set*  $S$  is the intersection of the closure of  $S$  and the closure of its complement. We use  $\partial S$  or  $\text{bdy } S$  to indicate the boundary of  $S$ . Thus  $\partial S = \bar{S} \cap \overline{S^c}$ .

The boundary of the open ball  $S = B_\varepsilon(\mathbf{y})$  is  $\partial S = \{\mathbf{x} : \|\mathbf{x} - \mathbf{y}\| = \varepsilon\}$ . We saw earlier that the closure of the ball includes all points with  $\|\mathbf{x} - \mathbf{y}\| \leq \varepsilon$ . The complement of the ball is closed, and consists of all points with  $\|\mathbf{x} - \mathbf{y}\| \geq \varepsilon$ . Taking the intersection yields  $\partial S = \{\mathbf{x} : \|\mathbf{x} - \mathbf{y}\| = \varepsilon\}$ .

If  $S = (a, b)$ , an interval in  $\mathbb{R}$ , the boundary consists of the two endpoints,  $\partial S = \{a, b\}$ . To see this, the closure is  $\bar{S} = [a, b]$  and the complement, which is closed, is  $S^c = (-\infty, a] \cup [b, +\infty)$ . Their intersection is  $\partial S = \{a, b\}$ . A similar argument shows that the boundary of intervals such as  $(a, +\infty)$  is  $\{a\}$ .

The boundary is not exactly the edge of a set, although clear edges are generally part of the boundary. Boundaries can sometimes be a little surprising. Consider the boundary of the set of rational numbers,  $\mathbb{Q}$ . Every irrational number can be written as a limit of rational numbers, so  $\overline{\mathbb{Q}} = \mathbb{R}$ . Every rational number can be written as the limit of irrational numbers (e.g.,  $x_n = x + (\pi/n)$ ), so  $\overline{\mathbb{Q}^c} = \mathbb{R}$ . Then  $\partial \mathbb{Q} = \mathbb{R}$ .

Let's take another case where it might not be obvious what the boundary is. In  $\mathbb{R}$ , consider  $S = \{1, 1/2, 1/4, \dots\}$ . The boundary is  $S \cup \{0\}$ , which is also the closure. Here the closure of the complement is  $\mathbb{R}$  itself.

### 12.43 Closures, Interiors, and Boundaries

► **Example 12.43.1: Closures, Interiors, and Boundaries.** Informally, we think of the closure as adding the boundary and the interior removing it. However, that is sometimes misleading. You may not always be adding and removing the same boundary. Consider the set  $S = (0, 1) \cup (1, 2)$ . It has boundary  $\{0, 1, 2\}$ . The closure is  $[0, 2]$ , but if we take the interior, we don't get the original set. Instead, we get the interval  $(0, 2)$ . We also see this in the complement  $S^c = (-\infty, 0] \cup \{1\} \cup [2, \infty)$ , which has interior  $(-\infty, 0) \cup (2, \infty)$ . Closing that gives us  $(-\infty, 0] \cup [2, \infty)$  which is not  $S^c$ . ◀

**12.44 Closures and Complements I****SKIPPED**

In 1922, Kuratowski showed that you can potentially form up to 14 different sets by repeatedly taking the closure and complement of a set  $S$ .<sup>3</sup> This can even be done in the real line with the usual topology.

It's easy to get four. Let  $S = (0, 1)$ . Then  $\bar{S} = [0, 1]$ ,  $\bar{S}^c = (-\infty, 0) \cup (1, +\infty)$ , and  $\overline{(\bar{S}^c)} = (-\infty, 0] \cup [1, +\infty)$ . Taking the complement again brings us back to  $S$ . Starting with a half-open finite interval yields six sets.

As for 14, one example is

$$S = \{0\} \cup (1, 2) \cup (2, 3) \cup (\mathbb{Q} \cap (4, 5)).$$

The seven sets starting with the complement are

$$S^c = (-\infty, 0) \cup (0, 1] \cup \{2\} \cup [3, 4] \cup (\mathbb{Q}^c \cap (4, 5)) \cup [5, +\infty)$$

$$\bar{S}^c = (-\infty, 1] \cup \{2\} \cup [3, +\infty)$$

$$(\bar{S}^c)^c = (1, 2) \cup (2, 3)$$

$$\overline{(\bar{S}^c)^c} = [1, 3]$$

$$\overline{\overline{(\bar{S}^c)^c}}^c = (-\infty, 1) \cup (3, +\infty)$$

$$\overline{\overline{\overline{(\bar{S}^c)^c}}}^c = (-\infty, 1] \cup [3, +\infty)$$

$$\overline{\overline{\overline{\overline{(\bar{S}^c)^c}}}}^c = (1, 3).$$

<sup>3</sup> Kazimierz Kuratowski (1896–1980) was a Polish mathematician. Besides the 14 sets, he's known the Kuratowski closure axioms, which characterize the closure operation and his contributions to the theory of Polish spaces (separable completely metrizable topological spaces). In economics, he's known for the KKM and KKMS Lemmas (KKM = Knaster-Kuratowski-Mazurkiewicz,  $S$  = Shapley), which are often used to show the existence of general equilibrium. Kuratowski also proved Zorn's Lemma 13 years before Zorn. Some refer to it as the Kuratowski-Zorn Lemma.

**12.45 Closures and Complements II****SKIPPED**

There are six more starting from the closure:

$$\begin{aligned}\bar{S} &= \{0\} \cup [1, 3] \cup [4, 5] \\ \bar{S}^c &= (-\infty, 0) \cup (0, 1) \cup (3, 4) \cup (5, \infty) \\ \overline{\bar{S}^c} &= (-\infty, 1] \cup [3, 4] \cup [5, \infty) \\ \overline{(\bar{S}^c)^c} &= (1, 3) \cup (4, 5) \\ \overline{\overline{(\bar{S}^c)^c}} &= [1, 3] \cup [4, 5] \\ \overline{\overline{\overline{(\bar{S}^c)^c}}} &= (-\infty, 1) \cup (3, 4) \cup (5, +\infty)\end{aligned}$$

That makes 14 in all, including  $S$  itself.

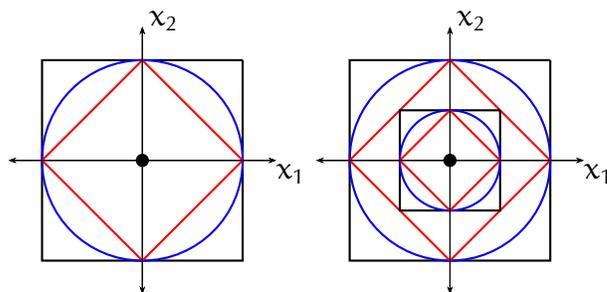
## 29.4 Equivalent Norms

Now that we know something about topologies, let's go back and explore the relation between a metric and its topology a little further. This material, which is a little more advanced, relates to Chapter 29.4 in S&B.

We know how to get the topology from a metric by using open balls. But can different metrics generate the same topology?

### 29.1 Nesting Balls

In fact, they can. In Theorem 12.13.1, we saw that both the product and  $\ell_2^m$  topologies had the same convergent sequences. In Theorem 29.6.1 we show that all of the  $\ell_p^m$  norms generate the same topology on  $\mathbb{R}^m$ .<sup>1</sup> The key is that you can nest balls for the various norms within one another, both up and down. So any  $\ell_p$  ball contains an  $\ell_q$  ball about each of its points, and vice-versa. That means that  $\ell_p^m$  open sets are  $\ell_q^m$  open, and vice-versa for  $1 \leq p, q \leq \infty$ .



**Figure 29.1.1:** The left panel shows balls of radius one in  $\ell_\infty$  (black),  $\ell_2$  (blue), and  $\ell_1$  (red) norms. The right panel shows how smaller balls of the same type nest inside them. Here the smaller balls have half the radius of the larger balls.

<sup>1</sup> See also section 29.4 of Simon and Blume.

## 29.2 Equivalent Norms

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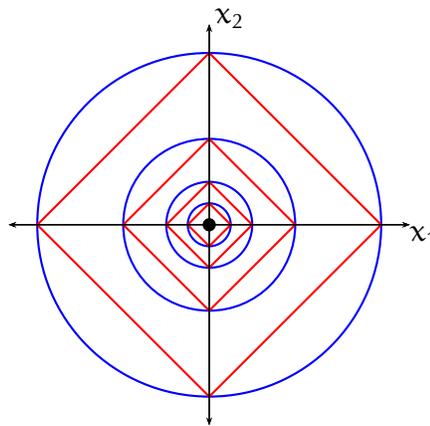
So let's look at the details about how different metrics can generate the same topology.

**Equivalent Norms.** On a vector space  $V$ , norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are *norm equivalent* or *equivalent norms* if there are positive numbers  $a$  and  $b$  with

$$a\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq b\|\mathbf{x}\|_2.$$

Of course,  $a \leq b$ . You may be bothered by the asymmetric treatment of the two norms. If so, good! The asymmetry means we need to do a bit of work to show that norm equivalence is an equivalence relation, putting each norm on an equal footing.

Norm equivalence means that the balls of each norm nest inside one another as illustrated in Figure 29.2.1 (see also Lemma 29.5.1).



**Figure 29.2.1:** The  $l_2$  (blue) and  $l_1$  (red) balls are illustrated, but at four different sizes, differing by factors of 2. See how they successively nest within one another. Each  $l_2$  ball has a smaller  $l_1$  ball inside it, and vice-versa

### 29.3 Norm Equivalence is an Equivalence Relation I

**Theorem 29.3.1.** For any vector space  $V$ , norm equivalence is an equivalence relation on the set of norms on  $V$ .

**Proof.** We need to show that norm equivalence is reflexive, symmetric, and transitive.

(1) The norm  $\|\cdot\|_1$  is equivalent to itself. Just set  $a = b = 1$  in the definition. So norm equivalence is **reflexive**.

(2) Suppose  $\|\cdot\|_1$  is equivalent to  $\|\cdot\|_2$ . Then there are  $a, b > 0$  with  $a\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq b\|\mathbf{x}\|_2$ . We take this apart into two inequalities and rewrite each piece.

$$a\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \quad \text{so} \quad \|\mathbf{x}\|_2 \leq \frac{1}{a}\|\mathbf{x}\|_1$$

and

$$\|\mathbf{x}\|_1 \leq b\|\mathbf{x}\|_2 \quad \text{so} \quad \frac{1}{b}\|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_2.$$

Put them back together to obtain

$$\frac{1}{b}\|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_2 \leq \frac{1}{a}\|\mathbf{x}\|_1,$$

showing that  $\|\cdot\|_2$  is equivalent to  $\|\cdot\|_1$ . This means norm equivalence is **symmetric**.

Proof continues ...

## 29.4 Norm Equivalence is an Equivalence Relation II

**Remainder of Proof.** (3) Now suppose  $\|\cdot\|_1$  is equivalent to  $\|\cdot\|_2$  and  $\|\cdot\|_2$  is equivalent to  $\|\cdot\|_3$ . Then there are  $a, b, c, d > 0$  with

$$a\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq b\|\mathbf{x}\|_2 \quad (29.4.1)$$

and

$$c\|\mathbf{x}\|_3 \leq \|\mathbf{x}\|_2 \leq d\|\mathbf{x}\|_3 \quad (29.4.2)$$

Again consider both halves of equation (29.4.2) separately. Multiply the left half by  $a$  and the right half by  $b$ , and reassemble with equation (29.4.1) in the middle to obtain

$$ac\|\mathbf{x}\|_3 \leq a\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq b\|\mathbf{x}\|_2 \leq bd\|\mathbf{x}\|_3$$

so

$$ac\|\mathbf{x}\|_3 \leq \|\mathbf{x}\|_1 \leq bd\|\mathbf{x}\|_3$$

This shows  $\|\cdot\|_1$  is equivalent to  $\|\cdot\|_3$ , that norm equivalence is **transitive**, and completes the proof. ■

## 29.5 The Nesting Ball Property

Equivalent norms have a nesting ball property as was illustrated in Figures 29.1.1 and 29.2.1.

**Lemma 29.5.1.** *If, for all  $\mathbf{x} \in V$ ,*

$$a\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq b\|\mathbf{x}\|_2.$$

*Then*

$$B_{\varepsilon/b}^2(\mathbf{x}) \subset B_{\varepsilon}^1(\mathbf{x}) \subset B_{\varepsilon/a}^2(\mathbf{x})$$

*where  $B_{\varepsilon}^i(\mathbf{x})$  is the  $\varepsilon$ -ball about  $\mathbf{x}$  computed using the  $i$ -norm.*

**Proof.** Suppose  $\mathbf{x} \in B_{\varepsilon}^1(\mathbf{x})$ . Then  $\|\mathbf{x}\|_1 < \varepsilon$ , so  $a\|\mathbf{x}\|_2 < \varepsilon$ , meaning that  $\|\mathbf{x}\|_2 < \varepsilon/a$ . It follows  $B_{\varepsilon}^1(\mathbf{x}) \subset B_{\varepsilon/a}^2(\mathbf{x})$ .

Now suppose  $\mathbf{x} \in B_{\varepsilon/b}^2(\mathbf{x})$ . Then  $\|\mathbf{x}\|_2 \leq \varepsilon/b$ , so  $\|\mathbf{x}\|_1 < b(\varepsilon/b) = \varepsilon$ . It follows that  $B_{\varepsilon/b}^2(\mathbf{x}) \subset B_{\varepsilon}^1(\mathbf{x})$ . ■

## 29.6 Equivalent Norms, Same Topology

We can use the nesting ball property to show that equivalent norms generate the same topology, the same collection of open sets.

**Theorem 29.6.1.** *Suppose two norms are equivalent on a vector space  $V$ . Then a set is open under norm  $\|\cdot\|_1$  if and only if it is open under  $\|\cdot\|_2$ .*

**Proof.** Suppose  $U$  is 1-open. Then for every  $\mathbf{x} \in U$ , there is an  $\varepsilon_x$  with  $B_{\varepsilon_x}^1(\mathbf{x}) \subset U$ . But then

$$B_{\varepsilon_x/b}^2(\mathbf{x}) \subset B_{\varepsilon_x}^1(\mathbf{x}) \subset U$$

by Lemma 29.5.1. That means  $U$  is 2-open because there is a 2-ball about each  $\mathbf{x} \in U$  that is contained in  $U$ .

Now suppose  $U$  is 2-open. Then for every  $\mathbf{x} \in U$ , there is an  $\varepsilon_x$  with  $B_{\varepsilon_x}^2(\mathbf{x}) \subset U$ . But then

$$B_{a\varepsilon_x}^1(\mathbf{x}) \subset B_{\varepsilon_x}^2(\mathbf{x}) \subset U$$

by Lemma 29.5.1 (nesting ball property). That means  $U$  is 1-open because there is a 1-ball about each  $\mathbf{x} \in U$  that is contained in  $U$ . ■

**29.7 The  $\ell_p$  Norms are Equivalent**

We can also show that any of the  $\ell_p$  norms are equivalent on  $\mathbb{R}^m$ .

**Theorem 29.7.1.** *For any  $p$ ,  $1 \leq p < \infty$ , the  $\ell_p$  and  $\ell_\infty$  norms are equivalent.*

**Proof.** First,

$$\|\mathbf{x}\|_p = \left( \sum_{i=1}^m |\mathbf{x}_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^m \|\mathbf{x}\|_\infty^p \right)^{1/p} = m^{1/p} \|\mathbf{x}\|_\infty.$$

Second,

$$\|\mathbf{x}\|_\infty^p \leq \sum_{i=1}^m |\mathbf{x}_i|^p,$$

so  $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_p$ , implying  $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_p$ .

Combining the results shows they are equivalent as

$$\|\mathbf{x}\|_p \leq m^{1/p} \|\mathbf{x}\|_\infty \leq m^{1/p} \|\mathbf{x}\|_p.$$

■

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**29.8 The  $\ell_p$  Topologies are Equivalent**

Combining the last two theorems shows that all of the  $\ell_p$  norms define the same open sets, and so yield the same topology. Moreover, that topology is equivalent to the product topology by Theorem 12.13.1.

**Theorem 29.8.1.** *For a given  $m$ , the open sets in  $\mathbb{R}^m$  are the same for every  $\ell_p$  norm,  $1 \leq p \leq \infty$ . The open sets are also the same in the product topology.*

**Proof.** For the first part, combine Theorems 29.7.1 and 29.6.1. Theorem 12.13.1 shows that a sequence product converges if and only if it converges in  $\ell_2$ . This implies that both topologies have the same closed sets, and by taking complements, the same open sets. ■

November 10, 2022