13. Functions of Several Variables

We previously covered some of the material in Simon and Blume's Chapter 13, which explains why we don't start the numbers at 13.1. We won't cover sections 13.1 and 13.2 from S&B, but we will examine continuity in rather more detail than they do in section 13.4. We will start with continuity in metric spaces, look at a number of examples, and then consider continuity in general topological spaces. It turns out that the general definition will be useful in metric spaces too.

For functions on the real line, a casual definition of continuity is that a function is continuous if you can draw its graph without lifting your pen from the paper. Of course, we will need a more formal definition.

13.12 Continuous Functions in Metric Spaces

We can use convergent sequences to define continuity in metric spaces.

Continuous Functions. Let f map the metric space (X, d_1) into a metric space (Y, d_2). A function f is *continuous at* x if $f(x_n) \xrightarrow{d_2} f(x)$ whenever $x_n \xrightarrow{d_1} x$. We say f is *continuous* if it is continuous at every point in X.

One needs to be a little careful here as there are two metrics involved, one on X, the other on Y. Here $x_n \xrightarrow{d_1} x$ means that $d_1(x_n, x) \to 0$ as $n \to \infty$, while $f(x_n) \xrightarrow{d_2} f(x)$ means that $d_2\big(f(x_n), f(x)\big) \to 0$ as $n \to \infty$.

13.13 Coordinate Functions are Continuous

One easy continuous function is the identity function given by the formula $\mathrm{id}(\boldsymbol{x}) = \boldsymbol{x}$.

▶ **Example 13.13.1: The Identity Function is Continuous.** We show that the identity function $\mathrm{id}(\boldsymbol{x}) = \boldsymbol{x}$ is continuous at any $\boldsymbol{x} \in X$ by taking any sequence with $\boldsymbol{x}_n \to \boldsymbol{x}$. Then

$$\mathrm{id}(\boldsymbol{x}_n) = \boldsymbol{x}_n \to \boldsymbol{x} = \mathrm{id}(\boldsymbol{x}),$$

showing that id is continuous at $\boldsymbol{x}$. Since $\boldsymbol{x}$ was an arbitrary point in X, id is continuous on X. ◀

Consider the normed space ℓ_p^m. It's not hard to show continuity of each of the coordinate functions $x_i(\boldsymbol{x}) = x_i$.

▶ **Example 13.13.2: Each Coordinate Function is Continuous in ℓ_p^m.** In ℓ_2^m, we already showed that $\boldsymbol{x}^n \to \boldsymbol{x}$ implies each coordinate converges (Theorem 12.13.1). But then Theorem 29.8.1 tells us convergence in all of the ℓ_p^m topologies is the same, and we are done. ◀

If $X = X_1 \times X_2 \times \cdots \times X_k$ is a product of metric spaces, the coordinate functions are also continuous. This follows immediately from the definition of product convergence. Product convergence, $\boldsymbol{x}^n \xrightarrow{p} \boldsymbol{x}$, means that $x_i^n \to x_i$ for every $i = 1, \dots, k$.

13.14 Example: Discontinuity at a Single Point

Not all functions are continuous. One simple example involves discontinuity at a point, where we have changed a continuous function at a single point to make it discontinuous. Thus

$$f(x) = \begin{cases} 0 & \text{when } x \neq 2 \\ 50 & \text{when } x = 2 \end{cases}$$

is discontinuous. To see this, let $x_n = 2 + 1/n$, so $x_n \to 2$.

$$\lim_{n\to\infty} f(x_n) = 0 \neq 50 = f(5).$$

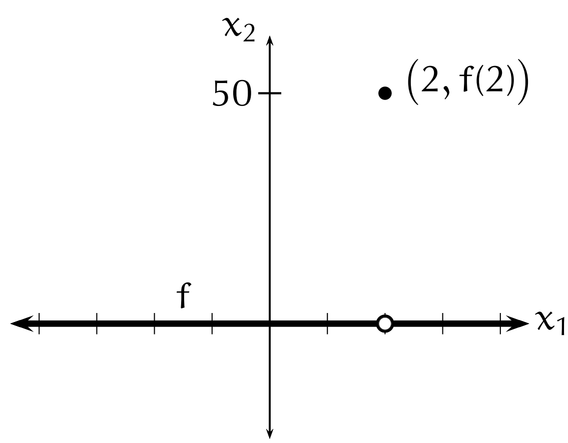

Figure 13.14.1: The function f is zero when $x \neq 2$ and 50 at $x = 2$. This causes a discontinuity at $x = 2$.

In both real and complex analysis, a function f is called *analytic* if it has a *power series representation*,

$$f(\mathbf{x}) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Functions which can be converted to an analytic function by redefining it at x_0, are said to have a *removable singularity*. This applies to our example above.

13.15 Example: Jump Discontinuity

▶ Example 13.15.1: Jump Discontinuity. The function g defined by

$$g(x) = \begin{cases} 0 & \text{when } x < 0 \\ 1 & \text{when } x \geq 0 \end{cases}.$$

is not continuous. If $x_n = -1/n$, then $x_n \to 0$, but $\lim_n g(x_n) = 0$, which is not $g(0) = 1$. The function is not continuous at $x = 0$.

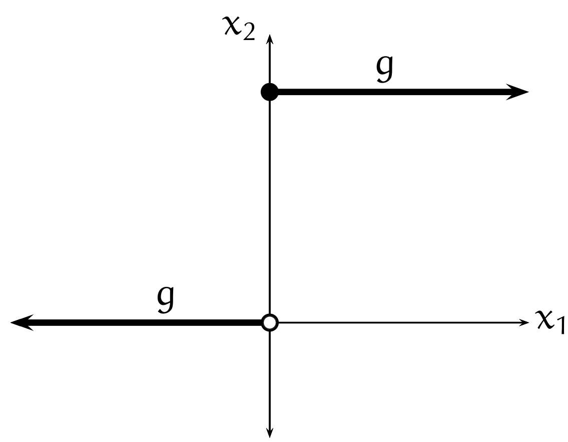

Figure 13.15.2: The function g is zero when $x \leq 0$ and 1 when $x > 0$. This causes a jump discontinuity at $x = 0$.

◀

13.16 Infinitely Many Isolated Jumps

Functions may have many jump discontinuities. They may even have have infinitely many jumps. We will continue to use the function g defined in Example 13.15.1.

▶ **Example 13.16.1: Infinitely Many Discrete Jumps.** The function $g(x-x_0)$ has the jump at x_0 instead of 0. The function

$$f(x) = \sum_{k=0}^{\infty} g(x-k)$$

has discontinuities at every non-negative integer. We don't have to worry about using an infinite sum because for any finite x there are only finitely many non-zero terms. In fact, if $x < n$, there are at most n non-zero terms in the sum. ◀

13.17 Example: A Pole Singularity

The previous examples involved jump discontinuities. There are other kinds of discontinuities.

▶ Example 13.17.1: Singularity. The function

$$f(x) = \begin{cases} 1/x & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

has a discontinuity at zero. In fact, it has a discontinuity at zero regardless of how we define $f(0)$.

This type of discontinuity is called a *singularity*, a term generally applied in real analysis to functions that have an infinite or undefined limit at some finite x_0. More precisely, this singularity is referred to as a *pole*. If a function has a *pole* at x_0, then $(x - x_0)^k f(x)$ will be continuous at x_0 for some m.[1] Here, $x^2 f(x) = x$ is continuous, even at $x_0 = 0$. The term "singularity" can also be applied when limits of derivatives are ill-behaved.

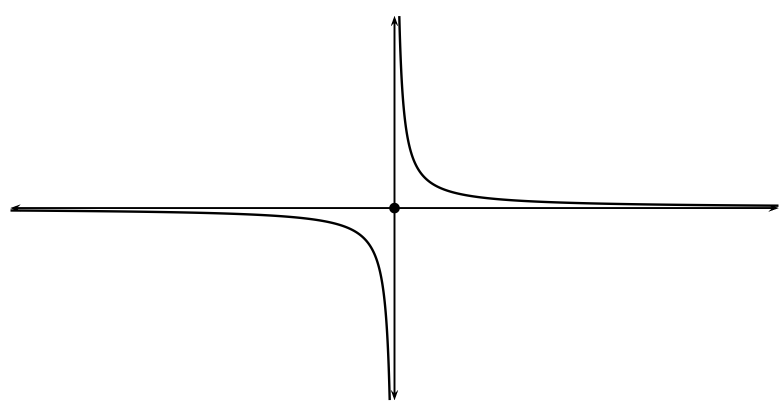

Figure 13.17.2: A Simple Pole at 0.

◀

[1] The actual definition of a pole of is that $(x - x_0)^k f(x)$ is analytic for some k, but we won't cover analytic functions in any detail in this course.

13.18 Example: An Essential Singularity

The following function exhibits another kind of discontinuity, an essential singularity.

► Example 13.18.1: Essential Singularity.

$$g(x) = \begin{cases} 0 & \text{when } x = 0 \\ \sin\left(\dfrac{1}{x}\right) & \text{when } x \neq 0. \end{cases}$$

The function g is a type of *topologist's sine curve*. To see that the function is discontinuous, consider the sequence $x_n = 2/n\pi$. Then $g(x_n) = \sin n\pi/2$, which successively takes the values $+1, 0, -1, 0, +1, \ldots$. It simply doesn't converge. It follows that $\lim_{x\to 0} g(x)$ doesn't exist, so g cannot be continuous.

In fact, can find sequences converging to zero where $\lim_n g(x_n)$ takes any value in $[-1, +1]$. This is an example of an *essential singularity*.

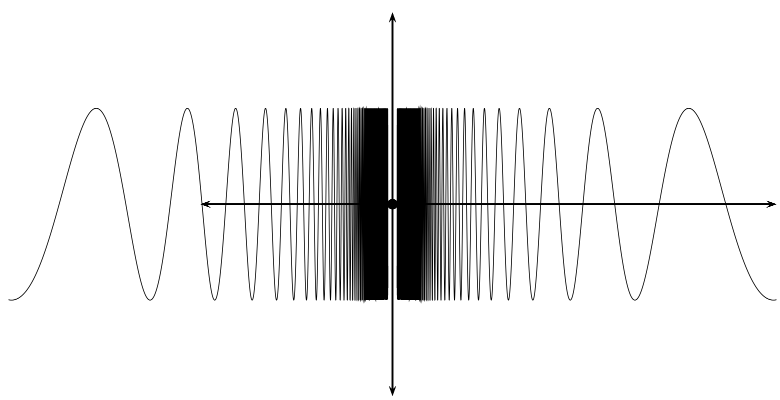

Figure 13.18.2: A Topologist's Sine Curve

◄

13.19 What Makes a Singularity Essential?

For $x \neq 0$, this function has the Laurent series representation[2]

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,x^{2n+1}} = \frac{1}{x} - \frac{1}{3!\,x^3} + \frac{1}{5!\,x^5} - \frac{1}{7!\,x^7} + \cdots .$$

A *Laurent series* is a power series that allows both positive and negative integer exponents with the general form

$$\sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n.$$

To understand better what an essential singularity is, we consider this as a function from the complex plane $\mathbb{C}$ to itself. For this, we write

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,z^{2n+1}},$$

which converges to an analytic function for all $z \neq 0$.

The Great Picard Theorem, which we state without proof, tells us how strange this function's behavior is around the origin, $z = 0$.[3]

Great Picard Theorem. *If a function* f *is analytic on* $\{z \in \mathbb{C} : |z - w| > 0\}$ *and has an essential singularity at* $z = w$*, then for any* $\varepsilon > 0$*, the function* f *takes on all complex values (or all but one) infinitely often on the punctured disk* $\{z : |z - w| < \varepsilon, z \neq w\}$*.*

[2] The French mathematician Pierre Alphonse Laurent (1813–1854) is best known for discovering the Laurent series, a generalization of Taylor series. He submitted his result for the Grand Prize of the Académie des Sciences in 1843, but it arrived too late for consideration. Apparently Weierstrass had already discovered this in 1841, but it was another half-century before his paper was published in his collected works.

[3] The Little Picard Theorem is also concerned with singularities. Émile Picard (1856–1941) was a French mathematician who worked in a wide variety of mathematical fields. We will later encounter the Picard Existence Theorem (Picard-Lindelöf Theorem) for differential equations.

13.20 Vector Operations are Continuous

When our metric space is a normed vector space $\big((V, +, \cdot), \|\cdot\|\big)$, we can ask about the continuity of the vector space operations: vector addition and scalar multiplication. Both of these are continuous. Further, the norm is continuous, and if V is an inner product space, the inner product is also continuous. We will cover each of these in turn over the next several pages.

We'll state the results first, and the proofs will follow over the next four pages. We start with continuity of the norm.

Theorem 13.21.1. *Let $\big(V, \|\cdot\|\big)$ be a normed vector space. Then the norm is a continuous function.*

Vector addition is continuous.

Theorem 13.23.1. *Let $\big(V, \|\cdot\|\big)$ be a normed vector space. Then vector addition is continuous.*

Scalar multiplication is also continuous

Theorem 13.24.1. *Let $\big(V, \|\cdot\|\big)$ be a normed vector space. Then scalar multiplication is continuous.*

If V is an inner product space, the inner product is also continuous.

Theorem 13.27.1. *Let $(V, \cdot\,)$ be an inner product space. Then $(\mathbf{x}, \mathbf{y}) \to \mathbf{x}\cdot\mathbf{y}$ is continuous.*

A series of remarks after the proofs explains some of the standard tricks and techniques that can help us show convergence and continuity.

13.21 The Norm is Continuous

In normed spaces we can consider the limit of the norm itself. Not surprisingly, the limit of the norm is the norm of the limit, making the norm continuous.

Theorem 13.21.1. *Let* $(V, \|\cdot\|)$ *be a normed vector space. Then the norm is a continuous function.*

Proof. We need to show that if $\{x_n\}$ converges to x then $\lim_n \|x_n\| = \|x\|$.

We use the triangle inequality twice to show

$$\|x\| \leq \|x_n\| + \|x - x_n\| \quad \text{and} \quad \|x_n\| \leq \|x\| + \|x - x_n\|.$$

meaning

$$\|x\| - \|x_n\| \leq \|x - x_n\| \quad \text{and} \quad \|x_n\| - \|x\| \leq \|x - x_n\|.$$

Together they imply that

$$\big|\|x_n\| - \|x\|\big| \leq \|x_n - x\| < \varepsilon$$

Now let $\varepsilon > 0$. We can find N with $\|x_n - x\| < \varepsilon$ for $n \geq N$. It follows that

$$\big|\|x_n\| - \|x\|\big| < \varepsilon$$

for $n \geq N$, so $\|x_n\| \to \|x\|$. Therefore the norm is continuous. ∎

13.22 Convergent Norms are Bounded

There is a useful corollary that states that the sequence $\{\|\mathbf{x}_n\|\}$ is bounded. We call this corollary a lemma because it is used in other proofs.

Lemma 13.22.1. *Let* $\{\mathbf{x}_n\}$ *converge to* $\mathbf{x}$ *in a normed vector space* $(V, \|\cdot\|)$. *Then there is a* $B \geq 1$ *with* $\|\mathbf{x}_n\| \leq B$ *for all* $n = 1, 2, \ldots$.

Proof. Since $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$, we can set $\varepsilon = 1$ and find an N so that

$$\big|\|\mathbf{x}_n\| - \|\mathbf{x}\|\big| < 1$$

for $n \geq N$. It follows that

$$\|\mathbf{x}_n\| \leq \big(\|\mathbf{x}_n\| - \|\mathbf{x}\|\big) + \|\mathbf{x}\| < 1 + \|\mathbf{x}\|$$

for $n \geq N$. That means that $1 + \|\mathbf{x}\|$ bounds $\|\mathbf{x}_n\|$ for every $n \geq N$. Then

$$\sup\big\{\|\mathbf{x}_1\|, \ldots\big\} \leq \max\big\{\|\mathbf{x}_1\|, \ldots, \|\mathbf{x}_{N-1}\|, 1 + \|\mathbf{x}\|\big\}.$$

Now set

$$B = \max\big\{\|\mathbf{x}_1\|, \ldots, \|\mathbf{x}_{N-1}\|, 1 + \|\mathbf{x}\|\big\} \geq 1 + \|\mathbf{x}\| \geq 1.$$

The maximum exists because we take the maximum over a finite set. ∎

13.23 Vector Addition is Continuous

We show the limit of a vector sum is the sum of the limit, proving continuity from $V \times V$ to V.

Theorem 13.23.1. *Let $(V, \|\cdot\|)$ be a normed vector space. Then vector addition is continuous.*

Proof. We need to show that if $\{x_n\}$ and $\{y_n\}$ are convergent sequences in V, with limits x and y, that $x_n + y_n \to x + y$

Let $\varepsilon > 0$. Since $x_n \to x$, we can choose N_1 with $\|x_n - x\| < \varepsilon/2$ for $n \geq N_1$. Then choose $N_2 \geq N_1$ with $\|y_n - y\| < \varepsilon/2$ for $n \geq N_2$.

It follows that whenever $n \geq N_2$,

$$\begin{aligned}
\left\|(x_n + y_n) - (x + y)\right\| &= \left\|(x_n - x) + (y_n - y)\right\| \\
&\leq \left\|x_n - x\right\| + \left\|y_n - y\right\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

This shows that $x_n + y_n \to x + y$. ∎

Triangle Inequality. We rearranged the terms on the top line to group the x and y terms. Then we used the triangle inequality to break the right-hand into two terms on the second line. In the process, we have created two terms that will involve ε. So we use $\varepsilon/2$ as a standard for each to ensure we end up with a single ε in the end.

This is not really necessary. Without it we would end up with 2ε on the right hand side. Since ε is any positive number, 2ε is also any positive number, and works just as well for showing convergence as ε.

What you need to avoid is having things such as n or N in the ultimate right-hand side. These can change with ε, perhaps in an ill-behaved fashion, creating unwanted infinities or indeterminate products such as $0 \times \infty$.

13.24 Scalar Multiplication is Continuous I

The other vector operation is scalar multiplication. The theorem says that the limit of the scalar product is the product of the limits.

Theorem 13.24.1. *Let $(V, \|\cdot\|)$ be a normed vector space. Then scalar multiplication is continuous.*

Proof. We need to show that if $\{x_n\}$ a convergent sequence in V with limit x, and $\alpha_n \to \alpha$ in $\mathbb{R}$. Then $\alpha_n x_n \to \alpha x$.

We start by considering the distance between $\alpha_n x_n$ and the proposed limit αx.

$$\begin{aligned} \|\alpha_n x_n - \alpha x\| &= \|\alpha_n x_n - \alpha x_n + \alpha x_n - \alpha x\| \\ &\le \|\alpha_n x_n - \alpha x_n\| + \|\alpha x_n - \alpha x\| \\ &= |\alpha_n - \alpha| \, \|x_n\| + |\alpha| \, \|x_n - x\| \\ &= |\alpha_n - \alpha| \, B + |\alpha| \, \|x_n - x\| \end{aligned}$$

In the last line we invoked Lemma 13.22.1 to find a $B \ge 1$ with $\|x_n\| \le B$ for every n. Let $\varepsilon > 0$ and choose N_1 so that

$$|\alpha_n - \alpha| < \frac{\varepsilon}{2B}$$

when $n \ge N_1$.

Proof continues ...

13.25 Scalar Multiplication is Continuous II

Remainder of Proof. Then choose $N_2 \geq N_1$ with

$$\|\mathbf{x}_n - \mathbf{x}\| < \frac{\varepsilon}{2\big(1 + |\alpha|\big)}$$

for $n \geq N_2$. The one in the denominator avoids any problems that might occur if $\alpha = 0$.

When $n \geq N_2$, both

$$|\alpha_n - \alpha| < \frac{\varepsilon}{2B} \quad \text{and} \quad \|\mathbf{x}_n - \mathbf{x}\| < \frac{\varepsilon}{2\big(1 + |\alpha|\big)}.$$

So for $n \geq N_2$,

$$\begin{aligned}
\|\alpha_n \mathbf{x}_n - \alpha \mathbf{x}\| &\leq |\alpha_n - \alpha|\, B + |\alpha| \|\mathbf{x}_n - \mathbf{x}\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon|\alpha|}{2\big(1 + |\alpha|\big)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

That proves the result. ∎

13.26 Secrets of Proof 13.24.1

Adding and Subtracting, Division by Zero. The proof used the old trick of adding and subtracting the same expression so that we can break things up using the triangle inequality, enabling us to deal separately with two simpler terms.

Another problem was the $\|\mathbf{x}_n\|$ term, which was bounded above by using Lemma 13.22.1.

In this case we have a more complicated situation with scaling ε in the other term. If we did not adjust it, we would have ended up with $\varepsilon|\alpha|$ instead of ε. Since $|\alpha|$ is independent of ε, the only potential problem is if we get zero.

In general, when rescaling ε, as we do in the proof, we can't afford to divide by something that might be zero, so we add one before dividing by $|\alpha|$ and by $\|\mathbf{x}\|$. This is also why we showed $B \geq 1$ in Lemma 13.22.1.

Bounding the Terms. This proof shows a new technique in addition to some we have seen earlier. There is a troublesome $\|\mathbf{x}_n\|$ term in the inequalities. The dependence on n means we can't attempt to make the companion term smaller than $\varepsilon/\big(1 + \|\mathbf{x}_n\|\big)$ as the target may converge to zero. This happens if $\|\mathbf{x}_n\|$ is unbounded.

We dealt with this by employing Lemma 13.22.1 to bound $\|\mathbf{x}_n\|$ from above.

13.27 Inner Products are Continuous

In inner product spaces, the inner product is continuous.

Theorem 13.27.1. *Let* $(V, \cdot)$ *be an inner product space. Then* $(x, y) \to x \cdot y$ *is continuous.*

Proof. We need to show that if $\{x_n\}$ and $\{y_n\}$ are convergent sequences with limits $\lim_n x_n = x$ and $\lim_n y_n = y$. Then $\lim_n x_n \cdot y_n = x \cdot y$.

Choose any $\varepsilon > 0$. We start by writing

$$\begin{aligned}
\left|x_n \cdot y_n - x \cdot y\right| &= \left|x_n \cdot y_n - x \cdot y_n + x \cdot y_n - x \cdot y\right| \\
&\le \left|x_n \cdot y_n - x \cdot y_n\right| + \left|x \cdot y_n - x \cdot y\right| \\
&\le \|y_n\|\|x_n - x\| + \|x\|\|y_n - y\| \\
&\le B\,\|x_n - x\| + \|x\|\|y_n - y\|
\end{aligned}$$

We used Lemma 13.22.1 in the last line to obtain a $B \ge 1$ with $\|y_n\| \le B$ for every n. Now choose N_1 with $\|x_n - x\| < \varepsilon/2B$ for $n \ge N_1$. Choose $N_2 \ge N_1$ with $\|y_n - y\| < \varepsilon/2\big(1 + \|x\|\big)$ for $n \ge N_2$.

Then for $n \ge N_2$,

$$\begin{aligned}
\left|x_n \cdot y_n - x \cdot y\right| &\le B\,\|x_n - x\| + \|x\|\|y_n - y\| \\
&\le B\frac{\varepsilon}{2B} + \|x\|\frac{\varepsilon}{2\big(1 + \|x\|\big)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

showing that $x_n \cdot y_n \to x \cdot y$, so the inner product is continuous. ∎

13.28 Continuous Functions, Topological Definition

We currently have three ways to describe a topology: open sets, closed sets, and for metric spaces only, convergent sequences. If we know any one of these we can derive the others. The real point is that any of these can be used to describe continuity.

Before giving the general definition, there is one more piece of notation to introduce. If $f\colon X \to Y$ is a function and the set $B \subset Y$, we define the *inverse image of* B, $f^{-1}(B)$, by $f^{-1}(B) = \{\mathbf{x} \in X : f(\mathbf{x}) \in B\}$.

Since the inverse image is defined in terms of **sets** rather than points, we cannot write it as a function from Y to X. Rather, $f^{-1}: \mathcal{P}(Y) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the *power set of* X, the set of all subsets of X. That is, f^{-1} maps subsets of Y to subsets of X.

Continuous Function (Topological Definition). A function $f\colon X \to Y$ is *continuous* if and only if $f^{-1}(U)$ is open whenever U is open.

13.29 Equivalent Definitions of Continuity I

In metric spaces, there are several conditions that are equivalent to continuity. Any of them can be used as the definition of continuity. Item (4) is Weierstrass's classic ε-δ definition, adapted to metric spaces.[4]

Theorem 13.29.1. *Suppose* (X, d_1) *and* (Y, d_2) *are metric spaces and* $f\colon S \to Y$ *where* $S \subset X$. *The following are equivalent.*

1. f *is continuous in the metric sense. For every* $x \in S$, *whenever* $x_n \xrightarrow{d_1} x$, $f(x_n) \xrightarrow{d_2} f(x)$.
2. $f^{-1}(U)$ *is open whenever* U *is open.*
3. $f^{-1}(A)$ *is closed whenever* A *is closed.*
4. *For all* $\varepsilon > 0$ *and* $x \in S$ *there is a* $\delta > 0$ *such that* $d_2\big(f(y), f(x)\big) < \varepsilon$ *whenever* $d_1(y, x) < \delta$.

[4] Karl Weierstrass (1815–1897) has been called the "father of modern analysis". He did more to rebuild the foundations of analysis in the 19^{th} century than anyone else (although Augustin-Louis Cauchy (1789–1857) did much to clear the ground for him). When you see an ε-δ argument, you should think of Weierstrass, even though Cauchy was the first to write such an argument. Weierstrass developed the ε-δ definition of continuity in 1861. He also was the first to publish an example of a nowhere differentiable continuous function, something that many mathematicians had previously believed impossible. The existence of such functions helped motivate the rebuilding project.

13.30 Equivalent Definitions of Continuity II 10/11/22

New Homework: Problems 29.3, 29.9, 29.11, and 29.13 are due on **Tuesday, October 18**.

Theorem 13.29.1. *Suppose* (X, d_1) *and* (Y, d_2) *are metric spaces and* $f\colon S \to Y$ *where* $S \subset X$. *The following are equivalent.*

1. f *is continuous in the metric sense. For every* $x \in S$, *whenever* $x_n \xrightarrow{d_1} x$, $f(x_n) \xrightarrow{d_2} f(x)$.
2. $f^{-1}(U)$ *is open whenever* U *is open.*
3. $f^{-1}(A)$ *is closed whenever* A *is closed.*
4. *For all* $\varepsilon > 0$ *and* $x \in S$ *there is a* $\delta > 0$ *such that* $d_2\big(f(y), f(x)\big) < \varepsilon$ *whenever* $d_1(y, x) < \delta$.

Proof. (1) implies (2). By way of contradiction, **suppose** $f^{-1}(U)$ **is not open**. Then there is a $x \in f^{-1}(U)$ where every $B_{1/n}(x)$ contains a point not in $f^{-1}(U)$. We can take $x_n \in B_{1/n}(x)$ with $f(x_n) \notin U$. Because $x_n \in B_{1/n}(x)$, $x_n \xrightarrow{d_1} x$. Now f is continuous, so $f(x_n) \xrightarrow{d_2} f(x) \in U$. Since U is open, there is N with $f(x_n) \in U$ for $n \geq N$. But then, $x_n \in f^{-1}(U)$ for $n \geq N$, **contradicting.** $x_n \notin f^{-1}(U)$. Therefore $f^{-1}(U)$ must be open.

Proof continues ...

13.31 Equivalent Definitions of Continuity III

Theorem 13.29.1. *Suppose* (X, d_1) *and* (Y, d_2) *are metric spaces and* $f\colon S \to Y$ *where* $S \subset X$*. The following are equivalent.*

1. f *is continuous in the metric sense. For every* $x \in S$*, whenever* $x_n \xrightarrow{d_1} x$*,* $f(x_n) \xrightarrow{d_2} f(x)$*.*
2. $f^{-1}(U)$ *is open whenever* U *is open.*
3. $f^{-1}(A)$ *is closed whenever* A *is closed.*
4. *For all* $\varepsilon > 0$ *and* $x \in S$ *there is a* $\delta > 0$ *such that* $d_2\big(f(y), f(x)\big) < \varepsilon$ *whenever* $d_1(y, x) < \delta$*.*

Remainder of Proof. (2) if and only if (3). Since $f^{-1}(A^c) = \big[f^{-1}(A)\big]^c$, and A is closed if and only if A^c is open, parts (2) and (3) are equivalent.

(2) implies (4). Now $B_\varepsilon\big(f(x)\big)$ is a open set, so $f^{-1}\big(B_\varepsilon\big(f(x)\big)\big)$ is also open and contains x. It follows that there is a $\delta > 0$ with $B_\delta(x) \subset f^{-1}\big(B_\varepsilon\big(f(x)\big)\big)$, proving (4).

(4) implies (1). Let $\varepsilon > 0$. We can choose $\delta > 0$ so that $f\big(B_\delta(x)\big) \subset B_\varepsilon\big(f(x)\big)$. Now take N so that $x_n \in B_\delta(x)$ whenever $n \geq N$. It follows that $f(x_n) \in B_\varepsilon\big(f(x)\big)$ for $n \geq N$, showing that f is continuous at x. Since $x \in S$ was arbitrary, f is continuous.

The circle (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (1) shows that (1), (2), and (4) are equivalent. The fact that (2) $\Leftrightarrow$ (3) means that (3) is equivalent to the other three. ∎

13.32 What Happened to the Metrics?

Where are the Metrics Hiding?. If you examine the proof closely, you'll find that each of d_1 and d_2 appears only once, when showing (1) implies (2). How can that happen? We're proving something that depends on which metric we use, so why do they only appear once?

The secret is in the definitions of the ε-balls and δ-balls. Some balls are constructed using d_1, others using d_2. That this information is hidden is harmless here, but may not always be so. You might want to identify which ball is which, perhaps by marking them with superscripts 1 and 2 and convince yourself there is no problem involving them.

13.33 ε–δ Continuity

Here's an example using condition (4) to show continuity. It's pretty similar to how we usually show continuity in metric spaces.

▶ Example 13.33.1: A Quadratic Continuous Function. Let $f(x) = x^2$. Let $\varepsilon > 0$. Choose a positive $\delta < \min\{1, \varepsilon/(2|x| + 1)\}$. It follows that for $y \in B_\delta(x)$,

$$\begin{aligned} \big|f(x) - f(y)\big| &= \big|x^2 - y^2\big| \\ &= \big|(x-y)(x+y)\big| \\ &= \big|x-y\big|\,\big|x+y\big| \\ &\le \big|x-y\big|\,\big(2|x|+1\big) \\ &\le \delta\big(2|x|+1\big) \\ &< \varepsilon. \end{aligned} \tag{13.33.1}$$

Here we used the fact that if $|x - y| < \delta$,

$$|x+y| = |2x - (x-y)| \le 2|x| + |x-y| \le 2|x| + 1.$$

This allows us to obtain equation (13.33.1), which shows that $f(y) \in B_\varepsilon\big(f(x)\big)$ whenever $y \in B_\delta(x)$, so $f(x) = x^2$ is continuous at any $x \in \mathbb{R}$ by condition (4). ◀

13.34 Inverse Images are Closed

The following example uses condition (3), that the inverse image of closed sets is closed.

▶ **Example 13.34.1: Continuity, Weak Inequalities, and Half-Spaces.** Let $f\colon S \to \mathbb{R}$ be continuous on $S \subset \mathbb{R}^m$. Suppose $\mathbf{x}_n \to \mathbf{x}$ and $f(\mathbf{x}_n) \geq \alpha$. We can rewrite this as $\mathbf{x}_n \in f^{-1}[\alpha, +\infty)$. Since $[\alpha, +\infty)$ is closed, so is its inverse image, and we can conclude that $f(\lim_n \mathbf{x}_n) \geq \alpha$.

It follows that the half-spaces $H^+(\mathbf{p}, \alpha) = \{\mathbf{x} : \mathbf{p} \cdot \mathbf{x} \geq \alpha\}$ and $H^-(\mathbf{p}, \alpha)\{\mathbf{x} : \mathbf{p} \cdot \mathbf{x} \leq \alpha\}$ are both closed sets because $f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x}$ is continuous by Theorem 13.27.1 and $H^+(\mathbf{p}, \alpha) = f^{-1}\big([\alpha, +\infty)\big)$ while $H^-(\mathbf{p}, \alpha) = f^{-1}\big((-\infty, \alpha]\big)$. Both are the inverse images of closed sets.

Moreover, $H(\mathbf{p}, \alpha) = f^{-1}\big(\{\alpha\}\big)$, so the hyperplane $H(\mathbf{p}, \alpha)$ is also closed. Alternatively, we could use the fact that $H(\mathbf{p}, \alpha) = H^+(\mathbf{p}, \alpha) \cap H^-(\mathbf{p}, \alpha)$ to show the hyperplane is closed as the intersection of closed sets. ◀

13.35 Budget Sets are Closed

We can use the fact that the inverse image of a closed set is closed to show the budget set is closed for all values of $\mathbf{p}$ and m.

▶ Example 13.35.1: The Budget Set is Closed. Recall that the budget set is defined by

$$B(\mathbf{p}, m) = \{\mathbf{x} \in \mathbb{R}^m_+ : \mathbf{p}\cdot\mathbf{x} \leq m\}.$$

We can combine the previous example with Example 10.60.1 to see that the budget set is closed. In Example 10.60.1, we found that

$$B(\mathbf{p}, m) = H^-(\mathbf{p}, m) \bigcap \left(\bigcap_{i=1}^{n} H^+(\boldsymbol{e}_i, 0) \right).$$

Then $B(\mathbf{p}, m)$ is closed because it is the intersection of closed half-spaces. ◀

13.36 Combining Continuous Functions

There are a number of ways to make continuous functions from other continuous functions. Most of the standard arithmetic operations: addition, subtraction, multiplication, and division, are continuous operations, as long as you don't divide by zero. We already saw this even in normed spaces for addition, subtraction, and scalar multiplication in Theorems 13.23.1 and 13.24.1.

Another useful way of creating continuous functions from continuous functions is composition.

Theorem 13.36.1. *Suppose* $f \colon A \to B$ *and* $g \colon B \to C$ *are both continuous. Then* $g \circ f \colon A \to C$ *is continuous.*

Proof. Suppose U is open in C. Then $(g \circ f)^{-1}(U) = f^{-1}\big(g^{-1}(U)\big)$. Now $g^{-1}(U)$ is an open subset of B because g is continuous, and so $f^{-1}\big(g^{-1}(U)\big)$ is open because f is continuous. Then $g \circ f$ is continuous. ∎

Finally, both products and quotients are continuous, as we will show in Theorems 13.37.1 and 13.38.1.

As a consequence, any polynomial

$$p(x) = \sum_{i=0}^{n} a_i x^{n-i} = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

is continuous.

13.37 Products are Continuous

We can use Theorem 13.36.1 in an easy proof that products are continuous

Theorem 13.37.1. *Let $S \subset \mathbb{R}^m$. If $f\colon S \to \mathbb{R}$ is a continuous real-valued function and $\mathbf{g}\colon S \to \mathbb{R}^m$ is continuous, then $f \times \mathbf{g}$ is continuous.*

Proof. We will employ Theorem 13.36.1.

Consider the mapping $F\colon \mathbb{R} \times \mathbb{R}^m$ defined by $F(\alpha, \mathbf{x}) = \alpha\mathbf{x}$. We know this is continuous by Theorem 13.24.1. Now define the mapping $G\colon \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^m$ by $G(\mathbf{x}) = \big(f(\mathbf{x}), \mathbf{g}(\mathbf{x})\big)$. This is also continuous. It then follows from Theorem 13.36.1 that $F \circ G(\mathbf{x}) = f(\mathbf{x})\mathbf{g}(\mathbf{x})$ is continuous. ∎

It's important in the above proof that $F(\alpha, \mathbf{x})$ is jointly continuous, not separately continuous. By *jointly continuous*, we mean that whenever $(\alpha_n, \mathbf{x}_n) \to (\alpha, \mathbf{x})$, then $F(\alpha_n, \mathbf{x}_n) \to F(\alpha, \mathbf{x})$ as shown in Theorem 13.24.1. By *separately continuous*, we mean that if $\alpha_n \to \alpha$, then $F(\alpha_n, \mathbf{x}) \to F(\alpha, \mathbf{x})$ for each $\mathbf{x}$ and if $\mathbf{x}_n \to \mathbf{x}$, then $F(\alpha, \mathbf{x}_n) \to F(\alpha, \mathbf{x})$ for each α.

13.38 Quotients are Continuous

Products and quotients of functions are continuous too, provided they make sense (no division by zero).

Theorem 13.38.1. *The function* $1/x$ *is continuous on* $(0, +\infty)$.

Proof. Suppose $x, x_n > 0$ and $x_n \to x$. Then

$$\left|\frac{1}{x} - \frac{1}{x_n}\right| = \left|\frac{x_n - x}{x x_n}\right| \tag{13.38.2}$$

Choose N_1 so that $|x_n - x| < x/2$ for $n \geq N_1$. Then $x - x_n < x/2$, implying $x/2 < x_n$. It follows that $1/x_n < 2/x$ for $n \geq N_1$. Now choose $N_2 \geq N_1$ with $|x_n - x| < \varepsilon x^2/2$. Substituting in equation (13.38.2), we obtain

$$\left|\frac{1}{x} - \frac{1}{x_n}\right| = \left|\frac{x_n - x}{x x_n}\right| < \frac{\varepsilon x^2}{2}\frac{1}{x x_n} = \frac{\varepsilon x}{2 x_n} < \frac{2\varepsilon x}{2x} < \varepsilon$$

for $n \geq N_2$. This shows that $x \mapsto 1/x$ is continuous on $(0, +\infty)$. ∎

Corollary 13.38.2. *If* $g\colon X \to (0, +\infty)$ *with* $X \subset \mathbb{R}^k$, *then* $f(\mathbf{x}) = 1/g(\mathbf{x})$ *is continuous on* X.

Proof. This follows from Theorems 13.38.1 and 13.36.1. ∎

Bounding the Inverse. We have a new challenge here connected with the fact that a $1/|x_n|$ term appears in the inequalities. This can blow up if $|x_n| \to 0$. The way to deal with is to show that $|x_n|$ must stay away from 0. In fact, we show it is eventually bounded below by the non-zero number $x/2$, so $1/|x_n|$ is bounded above by $2/x$.

13.39 Separately but Not Jointly Continuous Function I

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 0 & \text{when } (x,y) = (0,0) \\ \dfrac{xy}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \end{cases}$$

Properly defining this function at the origin is a problem. We have set $f(0,0) = 0$ in an attempt to remove the discontinuity. It does remove it for sequences of the form $(x_n, 0)$ and $(0, y_n)$, but fails to eliminate it. This gives us an interesting case where the function is continuous on lines parallel to the coordinate axis, but is not continuous from all directions.

Let's start with the case $(x_n, y) \to (0, y)$, where we take lines parallel to the x-axis. Suppose $x_n \to 0$. There are two subcases.

Subcase 1: If $y = 0$, $f(x_n, y) = 0$ for every x_n and so $\lim f(x_n, y) = 0$, showing continuity at $(0,0)$ as

Subcase 2: If $y \neq 0$. Then

$$f(x_n, y) = \frac{x_n y}{x_n^2 + y^2} \to 0 = f(0, y)$$

which is continuous at $(0, y)$

The other case, where we look at lines parallel to the y-axis give similar results: If $y_n \to 0$, $f(x, y_n) \to 0 = f(x, 0)$. Both cases together show that f is separately continuous in each variable.

13.40 Separately but Not Jointly Continuous Function II

But f is **not** jointly continuous. Let's take a sequence that converges diagonally to $\mathbf{0}$: $(x_n, y_n) = (1/n, 1/n)$. Now

$$\begin{aligned} f(x_n, y_n) &= \frac{1}{n^2} \cdot \frac{1}{n^{-2} + n^{-2}} \\ &= \frac{1}{n^2} \cdot \frac{n^2}{2} \\ &= 1/2 \end{aligned}$$

Then $\lim_n f(x_n, y_n) = 1/2 \neq 0 = f(0,0)$, so f is not jointly continuous at $(0,0)$. It is jointly continuous everywhere else.

Let's explore the discontinuity further. We know $2|xy| \leq x^2 + y^2$, so $|f(x,y)| \leq 1/2$.[5] The function is bounded, so the discontinuity cannot be too bad. Nonetheless, we can obtain the full range of outcomes if we take limits along lines of different slopes converging to $(0,0)$.

To see this, use polar coordinates and set $(x_n, y_n) = r_n(\cos\theta, \sin\theta)$ with $r_n > 0$ and $r_n \to 0$. If $\theta = 0$, this is convergence along the x-axis, while if $\theta = \pi/2$, it is convergence along the y-axis. Other choices of θ cover the other lines through the origin.

Given θ,

$$f(x_n, y_n) = \frac{\sin\theta\cos\theta}{\sin^2\theta + \cos^2\theta} = \frac{1}{2}\sin 2\theta.$$

This can take on any value between $-1/2$ and $1/2$.

[5] If you don't know this useful fact, consider that $(x+y)^2 \geq 0$ and $(x-y)^2 \geq 0$. Expanding each and moving the xy term to the right hand side, we obtain $x^2 + y^2 \geq -2xy$ and $x^2 + y^2 \geq 2xy$. Combined, these inequalities show that $x^2 + y^2 \geq 2|xy|$.

29. Limits and Compact Sets

This section covers various topics from and related to Chapter 29, sometimes with added material. These include boundedness, Dedekind completeness, monotone sequences, completeness, and compactness. By the end of this chapter, all but section 29.3, on connected sets, will be well covered. We'll get section 29.3 later.

29.9 Upper Bounds in $\mathbb{R}$

It's easier to think about boundedness in the real line. In the real line, a set is bounded if and only if there are A and B with $A \leq x \leq B$ for every $x \in S$. In other words, we have two bounds, a lower bound A and upper bound B.

The most important of all upper bounds is the smallest of them—the tightest bound. It is called the *least upper bound* or *supremum*.

Upper Bounds. Let $S \subset \mathbb{R}$. An *upper bound for* S is a number B with $x \leq B$ for all $x \in S$. The *least upper bound* or *supremum of* S is the smallest number C with $x \leq C$ for all $x \in S$.

We denote the least upper bound (supremum) of S by $\sup S$. Then if B is an upper bound for S, it obeys $B \geq \sup S$.

29.10 Examples of Upper Bounds

If $I = (a, b)$, $\sup I = b$ as b is an upper bound for the interval I. The numbers $b + 1$, $b + \frac{3}{2}$, and $b + 2\pi$ are also upper bounds. The number b is the smallest of these upper bounds, and in fact, there is no upper bound that is smaller than b, making it the supremum. To see the latter, suppose $c < b$. Then c cannot be an upper bound for (a, b) because $c + \frac{1}{2}(b - c)$ is in (a, b) and is also larger than c.

If $S = \{x \in \mathbb{Q} : x^2 < 2\}$, any rational number greater than $\sqrt{2}$ is a upper bound for S and no rational number smaller than $\sqrt{2}$ is an upper bound for S. To see the latter, if $y < \sqrt{2}$, write enough digits of $\sqrt{2}$ to get a rational number larger than y which is still in S. Then $\sup S = \sqrt{2}$, even though $\sqrt{2}$ is not in the set S.

Let $S = \{x \in \mathbb{Q}$: there is a circle with diameter d, circumference c and $c/d > x\}$. Here $\sup S = \pi \notin S$.

29.11 Dedekind Completeness

The following axiom (or one that performs a similar role) is part of the definition of the real numbers.[1]

Dedekind Completeness Axiom. *Every non-empty set S of real numbers that is bounded above has a supremum, $\sup S$, and that supremum is a real number.*

[1] The German mathematician Richard Dedekind (1831-1916) was an important part of the rebuilding of the foundations of mathematics that went on in the mid to late 19^{th} century. Besides his work in set theory, and axiomatizing the natural numbers, he proposed the Dedekind cut as a way of constructing the real numbers from the rational numbers. Among other things, it distinguishes the real numbers from the hyperreal numbers, aka non-standard real numbers that include infinitely large and infinitesimal numbers. The real numbers defined using Dedekind cuts form a continuum, without any gaps.

29.12 Dedekind Completeness and Archimedes NEW

Dedekind completeness can be used to show that the real numbers satisfy the Archimedean property.[2]

Archimedean Property. Suppose $x, y \in \mathbb{R}$ with $0 \leq x, y$ for $x, y \in S$. If $nx \leq y$ for all $n = 1, 2, \ldots$, then $x = 0$.

The Archimedean property rules out number systems involving infinitesimals, such as the hyperreal numbers (a.k.a. non-standard real numbers). The real numbers do not contain such numbers because they are Dedekind complete.[3]

Theorem 29.12.1. *The real numbers have the Archimedean property.*

Proof. Suppose $0 \leq x, y$ and $nx \leq y$ for all $n = 1, 2, \ldots$ and that $x > 0$. Consider the set $S = \{nx : n = 1, 2, \ldots\}$. This set has upper bound y, so by Dedekind completeness it has a supremum $\bar{y}$. Now $\bar{y} - x < \bar{y}$. Since $\bar{y}$ is the least upper bound of S, there is $mx \in S$ with $mx > \bar{y} - x$. But then $\bar{y} > (m + 1)x$, contradicting the fact that $\bar{y} = \sup\{nx : n = 1, 2, \ldots\}$. ∎

[2] Archimedes of Syracuse (ca.287–ca.212 BC) was a Greek mathematician, astronomer, engineer, and physicist. He was the greatest mathematician of antiquity. Some of his methods, involving infinitely small numbers and the method of exhaustion, came surprisingly close to calculus, almost 2 millenia before Newton. With them, he was able to calculate the volume of various solid shapes. He's also known for Archimedes principle, that the buoyant force on an object in a fluid is equal to the weight of the displaced fluid, which he is said to have discovered while taking a bath.

[3] The Archimedean property plays a role in Whitney's take on the use of real numbers for physical measurements. See H. Whitney (1968a), The Mathematics of Physical Quantities: Part I: Mathematical Models for Measurement, *Amer. Math. Monthly* **75**, 115–138, and in the following issue, H. Whitney (1968b), The Mathematics of Physical Quantities: Part II: Quantity Structures and Dimensional Analysis *Amer. Math. Monthly* **75**, 227–256.

29.13 Lower Bounds in $\mathbb{R}$

We can also consider lower bounds. Here too, one is most important, the *greatest lower bound* or *infimum*.

Lower Bounds. Let $S \subset \mathbb{R}$. A *lower bound for* S is a number D with $x \geq D$ for all $x \in S$. The *greatest lower bound* or *infimum of* S is the largest number C with $x \geq C$ for all $x \in S$. That is, if D is an lower bound for S, then $D \leq C$. We denote the greatest lower bound (infimum) of S by $\inf S$.

It's now easy to prove that sets in $\mathbb{R}$ that are bounded below have infima by using Dedekind completeness.

Theorem 29.13.1. *Every non-empty set S of real numbers that is bounded below has a infimum,* $\inf S$*. Moreover,* $\inf S$ *is a real number.*

Proof. Consider $-S = \{-x : x \in S\}$. It is bounded above and so has a supremum $\sup(-S)$ by Dedekind completeness. It is easy to show that $\inf S = -\big(\sup(-S)\big)$. ∎

29.14 More on Infima and Suprema

In $\mathbb{R}$ a set is *bounded* if and only if it has both upper and lower bounds.

If a non-empty set S is not bounded below, we set $\inf S = -\infty$. Similarly, a set that is not bounded above has $\sup S = +\infty$.

What about the empty set? If S is empty, $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. The rationale is that every number is a lower/upper bound for the empty set. This is vacuously true in that for every $a \in \mathbb{R}$, there is nothing in the empty set that is smaller or larger. Hence a is both an upper and lower bound. With all real numbers being both upper and lower bounds for the empty set, its greatest lower bound is $\inf \emptyset = \sup \mathbb{R} = +\infty$, while its least upper bound is $\sup \emptyset = \inf \mathbb{R} - \infty$.

When we allow the values $\pm\infty$ as suprema and infima, the Dedekind Completeness Axiom tells us that every set of real numbers has an infimum and a supremum, either of which may be infinite.

29.15 Bounded Sets

It makes sense to speak of bounded sets in a wider context. For this section we consider normed vector spaces $(V, \|\cdot\|)$ where it will make sense to define bounded sets.

Bounded. Let $(V, \|\cdot\|)$ be a normed vector space. A set $S \subset V$ is *bounded* if there is some number $K > 0$ with $\|\mathbf{x}\| \leq K$ for all $\mathbf{x} \in S$. If there is no such number, we say S is *unbounded*.

The definition is the same as saying that a set S is bounded if it is contained in some ball about zero. In other words, there is a $K > 0$ with $S \subset B_K(\mathbf{0})$.

In any ℓ_p^m, for any coordinate i, $|x_i| \leq \|\mathbf{x}\|_p$. As a result, $\|\mathbf{x}\|_p \leq K$ implies $|x_i| \leq K$ for $i = 1, \ldots, m$. In other words, bounded sets are bounded in each direction.

29.16 Unbounded Sets

Subsets of a normed vector space are necessarily either bounded or unbounded. If S is unbounded, we can find a sequence in S with $\|x_n\| \to +\infty$, meaning that for all K, there is an integer N with $\|x_n\| \geq K$ for all integers $n \geq N$.

Lemma 29.16.1. *If a set $S \subset V$ is unbounded where V is a normed space, then there are $x_n \in S$ with $\|x_n\| > n$.*

Proof. If S is not bounded, then for each positive integer n there will be a point $x_n \in S$ with $\|x_n\| > n$ as otherwise $S \subset B_n(0)$. ∎

29.17 Budget Sets are Bounded, Usually

One important result is that budget sets are bounded when prices are strictly positive.

► Example 29.17.1: Bounded Budget Sets. The budget set $B(\mathbf{p}, m) = \{\mathbf{x} \in \mathbb{R}^m_+ : \mathbf{p}\cdot\mathbf{x} \le m\}$ is bounded in any ℓ^m_p when $\mathbf{p} \gg \mathbf{0}$, but is not bounded if some $p_i = 0$.

When $\mathbf{p} \gg \mathbf{0}$ and $\mathbf{x} \in B(\mathbf{p}, m)$, $p_i x_i \le m$, so $0 \le x_i \le m/p_i$. Let $K_0 = \max_i\{m/p_i\}$.

In the case $p = \infty$,

$$\|\mathbf{x}\|_\infty \le K_0.$$

This implies the budget set is ℓ^m_∞ bounded.

In the case $1 \le p < \infty$,

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^m |x_i|^p\right)^{1/p} \le m^{1/p}K_0.$$

For $K_p = m^{1/p}K_0$, $\|\mathbf{x}\|_p \le K_p$, whenever $\mathbf{x} \in B(\mathbf{p}, m)$. This shows that the budget set is bounded in ℓ^m_p whenever $\mathbf{p} \gg \mathbf{0}$. ◄

29.18 A Bounded Budget Set

The boundedness of the budget sets with $\mathbf{p} \gg \mathbf{0}$ is illustrated in the figure below where the price vector is $\mathbf{p} = (1, 1)$, income is $m = 2.5$, and we use the ℓ_2-norm with the bound.

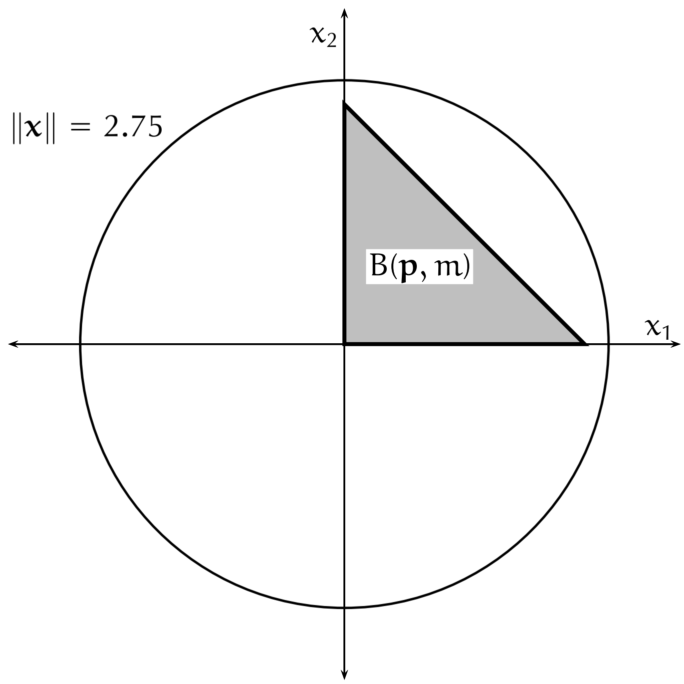

Figure 29.18.1: Here the price vector is $\mathbf{p} = (1, 1)$ and income is $m = 2.5$, resulting in a bounded budget set, as the enclosing circle with ℓ_2-norm 2.75 demonstrates.

29.19 An Unbounded Budget Set

Budget sets will be unbounded if one of the prices is zero.

▶ **Example 29.19.1: Unbounded Budget Set.** If some $p_i = 0$, we can increase x_i without bound and still stay in the budget set. So the budget set is not bounded. For example, in $\mathbb{R}^2$, set $\mathbf{p} = (1, 0)$ and $m = 1$. Then $B(\mathbf{p}, m) = \{\mathbf{x} \in \mathbb{R}^2_+ : 0 \le x_1 \le 1\}$ is unbounded, as illustrated in Figure 29.19.2.

Figure 29.19.2 makes clear that the budget set has no top, continuing onward to infinity in the coordinate whose price is zero. To make this a bit more concrete, consumption bundles such as $\mathbf{x}_n = (1, n)$ are in the budget set.

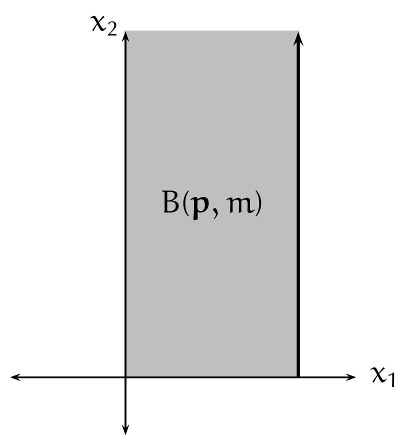

Figure 29.19.2: Here the price vector is $\mathbf{p} = (1, 0)$, resulting in a budget set that is unbounded above.

◀

29.20 Special Properties of Norm Distances

Distances derived from norms have two special properties that arbitrary metrics may not have.

The first is that the distance derived from the norm is translation invariant. If we add the same vector to two other vectors, it doesn't change the distance between them. The points x and y are the same distance apart as are $(x + z)$ and $(y + z)$. This follows from the way we define distance using a norm:

$$\begin{aligned} d(x, y) &= \|x - y\| \\ &= \|(x + z) - (y + z)\| \\ &= d(x + z, y + z). \end{aligned}$$

Distances derived from **norms are always translation invariant**.

The second property derives from the absolute homogeneity of degree one. Multiplying two vectors by the same scalar multiplies their distance by the absolute value of that scalar.

$$\begin{aligned} d(\lambda x, \lambda y) &= \|\lambda x - \lambda y\| \\ &= |\lambda|\,\|x - y\| \\ &= |\lambda|\, d(x, y). \end{aligned}$$

Distances derived from **norms scale as the vectors involved scale**.

29.21 Absorbing Sets

A set is absorbing if every point is contained in some multiple of it.

Absorbing Set. Let V be a vector space. A set $B \subset V$ is *absorbing* if for every $\mathbf{x} \in V$, there is an $\alpha > 0$ with $\mathbf{x} \in \beta B$ whenever $|\beta| \geq \alpha$.

It is easy to show that the open unit ball about zero in any normed space is an absorbing set.

Theorem 29.21.1. *Let $\big(V, \|\cdot\|\big)$ be a normed vector space and $B = \{\mathbf{x} \in V : \|\mathbf{x}\| < 1\}$ be the open unit ball about zero. Then B is an absorbing set.*

Proof. Let $\mathbf{x} \in V$. Let $\alpha = \|\mathbf{x}\| + 1$. Then for $|\beta| > \alpha$, $\mathbf{x} \in \beta B = B_\beta(\mathbf{0})$ since $\|\mathbf{x}\| < |\beta|$. ∎

29.22 The Unit Ball Defines the Norm

It turns out that if we know the unit ball, we can compute the norm of any vector using absolute homogeneity and then compute any distance by using translation invariance. Together, the unit ball gives us the information we need to infer any distances in the whole space.

Theorem 29.22.1. *Let* $(V, \|\cdot\|)$ *be a normed vector space and* $B = \{x \in V : \|x\| < 1\}$ *be the open unit ball. Then* $\|x\| = \inf\{\alpha > 0 : x \in \alpha B\}$*.*

Proof. Let $x \in V$. Now the set $\{\alpha > 0 : x \in \alpha B\}$ is non-empty since B is absorbing. Since it is bounded below by 0, it has an infimum $\bar{\alpha}$. We will show $\bar{\alpha} = \|x\|$.

If $\bar{\alpha} < \|x\|$, take α with $\bar{\alpha} < \alpha < \|x\|$. By definition of $\bar{\alpha}$, there is $x \in \alpha B$, so there is $y \in B$ with $x = \alpha y$. Then $\|x\| = |\alpha|\,\|y\| < \alpha$, contradicting $\alpha < \|x\|$. This shows $\bar{\alpha} \geq \|x\|$.

Now suppose $\bar{\alpha} > \|x\|$. Consider

$$y = \frac{2}{\big(\bar{\alpha} + \|x\|\big)}x.$$

Then $\|y\| < 1$, so $y \in B$. But then $x \in \frac{1}{2}\big(\bar{\alpha} + \|x\|\big)B$. Now $\frac{1}{2}\big(\bar{\alpha} + \|x\|\big) < \bar{\alpha}$. Then $x \in \frac{1}{2}\big(\bar{\alpha} + \|x\|\big)B$ contradicts the fact that $\bar{\alpha}$ is the infimum of scalars α with $x \in \alpha B$. The contradiction shows that $\bar{\alpha} \leq \|x\|$.

Combining our results yields $\|x\| = \bar{\alpha}$, as desired. ∎

29.23 Boundedness in Metric Spaces

If we try to define boundedness in metric spaces by bounding the metric, we may get sets that are quite unbounded in any ordinary (or mathematical) sense of the word. In particular, they may be unbounded in some directions, meaning that there is $\mathbf{x} \neq \mathbf{0}$ so that $n\mathbf{x}$ is in the set for all large n.

Here's an example.

▶ **Example 29.23.1: Unbounded Boundedness in a Metric Space.** Consider $\mathbb{R}^2$ with the metric used in Figure 10.53.1:

$$d(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\left(\frac{|x_1 - y_1|}{1 + |x_1 - y_1|}\right) + \frac{1}{4}\left(\frac{|x_2 - y_2|}{1 + |x_2 - y_2|}\right)$$

Suppose we have a set S where $d(\mathbf{0}, \mathbf{x}) \leq 1/4$ for all $\mathbf{x} \in S$. In spite of the bound on the metric, this set is not bounded!

In fact, $d\big(\mathbf{0}, (0, y_2)\big) = |y_2|/4\big(1 + |y_2|\big) < 1/4$, so the entire vertical axis is part of S. See also Figure 29.23.2 below.[4]

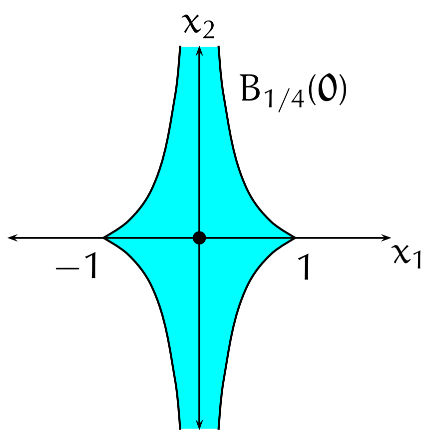

Figure 29.23.2: Although the sequence metric is bounded, that cannot be said about the ball of radius $1/4$ about $\mathbf{0}$. We restrict the metric to $\mathbb{R}^2$ in the diagram. The cyan area illustrates points $\mathbf{x}$ with $d(\mathbf{x}, \mathbf{0}) = 1/4$ in $\mathbb{R}^2$. The entire vertical axis has $d(\mathbf{x}, \mathbf{0}) < 1/4$, with $d\big((0, x_2), \mathbf{0}\big) \to 1/4$ as $x_2 \to \pm\infty$.

◀

[4] You may recognize this set from Figure 10.53.1.

29.24 Monotone Sequences in $\mathbb{R}$

Sequences of real numbers that either never decrease, or never increase are called *monotone*.

Monotone Sequences. A sequence $\{x_n\}$ of real numbers is *monotone increasing* if $x_n \le x_{n+1}$ for $n = 1, 2, \ldots$. It is *monotone decreasing* if $x_n \ge x_{n+1}$ for $n = 1, 2, \ldots$. Saying a sequence is *monotone* means it is either monotone increasing or monotone decreasing.

Thus $x_n = 1/n$ is monotone decreasing and $x_n = n^2$ is monotone increasing.

Theorem 29.24.1. *Every bounded monotone sequence in $\mathbb{R}$ converges. Moreover, if x_n is increasing, $\lim_n x_n = \sup_n x_n$ and if x_n is decreasing, $\lim_n x_n = \inf_n x_n$.*

Proof. We will prove the decreasing case since Simon and Blume (Theorem 29.2) do the increasing case.

Let $x = \inf_n x_n$. I claim $x_n \to x$. Let $\varepsilon > 0$. Then $x + \varepsilon$ is not a lower bound for $\{x_n\}$, so there is N with $x \le x_N < x + \varepsilon$. Because x_n is monotone decreasing and x is a lower bound for the sequence, $x \le x_n \le x_N < x + \varepsilon$ for $n \ge N$. But then $|x_n - x| < \varepsilon$ for $n \ge N$ which shows that $\lim_n x_n = x$. ∎

The notations $x_n \downarrow x$ and $x_n \uparrow x$ are sometimes used to indicate monotone convergence to x.

29.25 Real Sequences Have Monotone Subsequences

We can extract a monotone subsequence from every sequence of real numbers.

Theorem 29.25.1. *Every sequence of real numbers has a monotone subsequence.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. **If** it has a monotone increasing subsequence, we are done.

Else, the sequence has **no** monotone increasing subsequence. Take x_i in the sequence. Then choose a $j > i$ with $x_j \geq x_i$, then a k with $k > j$ and $x_k \geq x_j$. Since there are no monotone increasing subsequences, this process ends after a finite number of steps (possibly immediately).

Call the highest number in the sequence a *dominant element*. Suppose it is x_ℓ. Then $x_\ell > x_n$ for all $n > \ell$. Then find the next dominant element following x_ℓ and repeat. This gives us a monotone decreasing subsequence of successive dominant elements and we are done. ∎

In other words, any sequence of real numbers has a monotone increasing subsequence, or a monotone decreasing subsequence, or both.

29.26 Cauchy Sequences

We still lack a criterion to tell if a sequence converges or not. At present, all we can do is try to find a limit. There is a test for convergence. Whether a sequence in $\mathbb{R}$ converges depends on whether it is a Cauchy sequence.

Cauchy Sequence. A sequence $\{x_n\}$ in a metric space (X, d) is a *Cauchy sequence* if for every $\varepsilon > 0$ there is an N with $d(x_n, x_m) < \varepsilon$ whenever $m, n \geq N$.

This means a sequence is a Cauchy sequence if the terms of the sequence get closer together.

29.27 Convergent Sequences are Cauchy

Any convergent sequences is a Cauchy sequence. The point is that the terms of the sequence get closer and closer to the limit, and so closer and closer to each other.

Theorem 29.27.1. *Suppose* $\{x_n\}$ *is a sequence in a metric space* (X, d). *If* $x_n \to x$, *then* $\{x_n\}$ *is a Cauchy sequence.*

Proof. Let $\varepsilon > 0$ be given. Since $x_n \to x$, there is an N with $d(x_n, x) < \varepsilon/2$ for $n \geq N$. Then for $m, n \geq N$,

$$d(x_m, x) < \varepsilon/2 \quad \text{and} \quad d(x_n, x) < \varepsilon/2.$$

It follows that

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \varepsilon$$

for $n, m \geq N$, showing that $\{x_n\}$ is a Cauchy sequence. ∎

29.28 Cauchy Sequences are Bounded

But is there a converse? Do Cauchy sequences converge? This is more difficult to prove for the simple reason that it is false in some metric spaces. In particular, it is false in the rational numbers $\mathbb{Q}$, where sequences that converge to irrational numbers such as $\sqrt{2}$ or π do not converge in $\mathbb{Q}$.

But what about $\mathbb{R}$? We start by showing Cauchy sequences in $\mathbb{R}$ are bounded.

Theorem 29.28.1. *Let $\left\{x_n\right\}$ be a Cauchy sequence in $\mathbb{R}$. Then $\{x_n\}$ is bounded.*

Proof. Set $\varepsilon = 1$ and choose N such that $|x_n - x_m| < 1$ for $n, m \geq N$. Then $|x_n - x_N| < 1$ for $n \geq N$. Now define

$$B = \max\left\{|x_n| : n \leq N\right\} + 1$$

Then B is a upper bound for $\{|x_n|\}$ since

$$\left|x_n\right| \leq \left|x_N\right| + \left|x_n - x_N\right| < \left|x_N\right| + 1 \leq B$$

for all $n \geq N$, and $|x_n| \leq B$ for $n \leq N$ by the definition of B. This shows that $|x_n|$ is bounded above, so x_n is bounded both above and below. ∎

29.29 Real Cauchy Sequences Converge

We can now use our previous theorems to show that every Cauchy sequence of real numbers converges.

Theorem 29.29.1. *Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$. Then $\{x_n\}$ converges.*

Proof. By Theorem 29.25.1, x_n has a monotone subsequence x_{n_k}. By Theorem 29.28.1, the subsequence is bounded, so by Theorem 29.24.1, the subsequence x_{n_k} converges to some $x \in \mathbb{R}$.

Let $\varepsilon > 0$. Choose N such that

$$\left|x_n - x_m\right| < \frac{\varepsilon}{2} \quad \text{whenever} \quad m, n \geq N.$$

Now find $n_k \geq N$ with $|x_{n_k} - x| < \varepsilon/2$. It follows that for all $m \geq N$,

$$\left|x - x_m\right| \leq \left|x - x_{n_k}\right| + \left|x_{n_k} - x_m\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

showing that $x_n \to x$. ∎

29.30 Complex Cauchy Sequences Converge 10/13/22

It's now easy to show that complex Cauchy sequences also converge.

Theorem 29.30.1. *Suppose z_n is a complex Cauchy sequence. Then there is $z \in \mathbb{C}$ with $z_n \to z$.*

Proof. Since $|\operatorname{Re} w| \leq |w|$ and $|\operatorname{Im} w| \leq |w|$ for any complex number w, the real and imaginary parts of z_n are real Cauchy sequences. As such, they converge to x and y. Let $z = x + iy$ Then $z_n \to z$. ∎

29.31 Complete Metric Spaces

We can now define complete metric spaces.[5]

Complete Metric Space. A metric space is *complete* if every Cauchy sequence converges.

The definition also applies to normed spaces because they are metric spaces. Theorem 29.29.1 showed that $\big(\mathbb{R}, |\cdot|\big)$ is a complete normed space, hence a complete metric space. Theorem 29.30.1 did the same for the complex numbers $\big(\mathbb{C}, |\cdot|\big)$.

[5] The term "complete" was introduced by French mathematician Maurice Fréchet (1878–1973) in his 1906 Ph.D. dissertation. He was the first to define metric spaces (although the name is due to Hausdorff), and one of the pioneers in functional analysis. He made a number of important contributions to topology, and to probability and statistics. He's best known for the Fréchet derivative.

29.32 Euclidean Spaces are Complete

It's not hard to show that $\mathbb{R}^m$ is complete in the Euclidean norm.

Theorem 29.32.1. *The normed space ℓ_2^m is complete.*

Proof. Let $\varepsilon > 0$. Choose N such that for $k, n \geq N$, $\|\mathbf{x}^k - \mathbf{x}^n\|_2 < \varepsilon$. Let x_i^k be the i^{th} component of $\mathbf{x}^k$. Then

$$\left|x_i^k - x_i^n\right| \leq \left\|\mathbf{x}^k - \mathbf{x}^n\right\|_2 < \varepsilon$$

for all $k, n \geq N$. It follows that each $\{x_i^n\}$ is a Cauchy sequence.

Then each $x_i^n \to x_i$ for some $x_i \in \mathbb{R}$. By Theorem 12.13.1, $\mathbf{x}^n \to \mathbf{x}$, showing that $\left(\mathbb{R}^m, \|\cdot\|_2\right)$ is complete. ∎

The same sequences converge in all of the ℓ_p norms, so each ℓ_p^m is complete.

Completeness and Dedekind Completeness. For the real numbers, completeness is equivalent to Dedekind completeness. It's possible to replace Dedekind completeness with completeness as an axiom for the reals. Then you can construct the reals from the rationals by using equivalence classes of Cauchy sequences instead of Dedekind cuts.

29.33 Complete Metric Vector Spaces

Several types of complete vector spaces have special names.

Special Spaces. A *Hilbert space* is an inner product space that is complete under the norm generated by the inner product. Thus ℓ_2^m is a Hilbert space, as are ℓ_2 and L^2.[6]

A *Banach space* is a normed space that is complete in the metric defined by the norm. It follows that any ℓ_p^m is a Banach space. As a matter of fact, so are ℓ_p and L^p. Any Hilbert spaces is also a Banach space.[7]

A vector space that is a complete metric space, with continuous linear operations, is called a *Fréchet space*. The sequence space $\mathbf{s}$ is a Fréchet space when using the metric defined in section 10.52. Any Banach or Hilbert space is also a Fréchet space.

Completeness requires a metric, or at least some sort of uniform structure that permits us to define Cauchy sequences. In vector spaces, one can use the vector structure to define such a uniformity. A *topological vector space* $\big((V,+,\cdot),\mathcal{T}\big)$ is a vector space $(V,+,\cdot)$, with a topology $\mathcal{T}$ that makes both vector addition and scalar multiplication continuous. It is possible to define completeness for many topological vector spaces.

[6] The German mathematician David Hilbert (1862–1943) was one of the leading turn-of-the-century mathematicians. He did fundamental research in many fields, ranging from number theory to mathematical physics. He's also known for his list of 23 unsolved problems, which influenced 20^{th} century mathematics. Some of them, such as the Riemann hypothesis, remain unsolved.

[7] The Polish mathematician Stefan Banach (1892–1945). He spent most of his in what was then Lwów, Poland (Lvov in German, now Lviv, Ukraine) and is best known for his work in functional analysis, including the Contraction Mapping Theorem.

29.34 Spaces of Continuous Functions

Another class of complete normed spaces are the spaces of bounded continuous functions on a set X with values in $\mathbb{R}^m$, with the supremum or uniform norm.

The Normed Space $\left(\mathcal{C}_b(X;\mathbb{R}^m), \|\cdot\|_\infty\right)$. The space of bounded continuous functions on X with values in $\mathbb{R}^m$ is

$$\mathcal{C}_b(X;\mathbb{R}^m) = \left\{\mathbf{f}\colon X \to \mathbb{R}^m : \mathbf{f} \text{ is bounded and continuous}\right\}.$$

It has the supremum (uniform) norm

$$\left\|\mathbf{f}\right\|_\infty = \sup\left\{|f_i(\mathbf{x})| : \mathbf{x} \in X, i = 1, \ldots, m\right\}$$

where f_i is the i^{th} component of $\mathbf{f}$. When $m = 1$, we may sometimes write $\mathcal{C}_b(X)$ instead of $\mathcal{C}_b(X;\mathbb{R}^m)$.

By saying $\mathbf{f}$ is bounded over X, we mean there is a $B > 0$ with $|f_i(\mathbf{x})| \leq B$ for every $i = 1, \ldots, m$ and $\mathbf{x} \in X$. By the Dedekind Completeness Axiom, $\|\mathbf{f}\|_\infty$ exists. Moreover, $\|\mathbf{f}\|_\infty \leq B$. Dedekind completeness allows us to define the norm as we did.

29.35 Pointwise Convergence

There are two types of convergence in $\mathcal{C}_b(X;\mathbb{R}^m)$. The first is pointwise convergence.

Pointwise Convergence. A sequence of functions $\mathbf{f}_n : X \to \mathbb{R}^m$ *converges pointwise to* $\mathbf{f}$ if $\lim_n \mathbf{f}_n(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for each $\mathbf{x} \in X$.

29.36 A Discontinuous Pointwise Limit

Unfortunately, the pointwise limit of continuous functions need not be continuous. The space of bounded continuous functions $\mathcal{C}_b(X;\mathbb{R}^m)$ is not pointwise complete.

► **Example 29.36.1: Discontinuous Limit of Continuous Functions.** Define $f_n\colon [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} nx & \text{for } x \in [0, 1/n)] \\ 1 & \text{for } x \in [1/n, 1]. \end{cases}$$

The f_n are continuous functions on $[0,1]$ because both definitions match at $x = 1/n$.

The pointwise limit is

$$f(x) = \begin{cases} 0 & \text{when } x = 0 \\ 1 & \text{for } x \in (0, 1]. \end{cases}$$

Of course, the limit function f is discontinuous at 0. It is illustrated in Figure 29.36.2. ◄

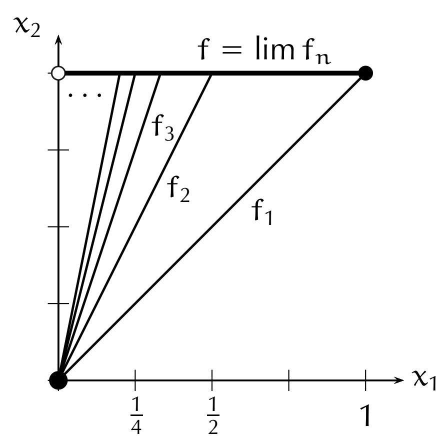

Figure 29.36.2: The functions f_n described in Example 29.36.1 converge upwards to f, which is the line at the top, except at zero, where it is zero, giving us a pointwise limit of continuous functions that is not continuous at zero.

One of the limitations of Riemann integration is that pointwise limits of Riemann integrable functions need not be Riemann integrable.

29.37 Uniform Convergence in $\mathcal{C}_b(X;\mathbb{R}^m)$

The second type of convergence is uniform convergence, convergence in the supremum norm.

Uniform Convergence. A sequence of functions $\mathbf{f}_n : X \to \mathbb{R}^m$ *converges uniformly to* $\mathbf{f}$ if $\|\mathbf{f}_n - \mathbf{f}\|_\infty \to 0$.

Unlike pointwise limits, uniform limits of functions in $\mathcal{C}_b(X;\mathbb{R}^m)$ are bounded continuous functions.

To avoid possible confusion, we need to distinguish two supremum norms, on $\mathbb{R}^m$ and on $\mathcal{C}_b(X;\mathbb{R}^m)$. We denote the supremum norm on $\mathbb{R}^m$ by $\|\mathbf{x}\|_{m,\infty} = \sup_i |x_i|$ and the supremum norm on $\mathcal{C}_b(X;\mathbb{R}^m)$ by $\|\mathbf{f}\|_\infty$. They are related by the equation

$$\|\mathbf{f}\|_\infty = \sup\left\{\|\mathbf{f}(\mathbf{x})\|_{m,\infty} : \mathbf{x} \in X\right\}$$

so

$$\|\mathbf{f}(\mathbf{x})\|_{m,\infty} \leq \|\mathbf{f}\|_\infty$$

for all $\mathbf{x} \in X$.

29.38 Uniform Limits of Continuous Functions I

One important result is that the uniform limit of continuous functions is a continuous function, something that cannot always be said about the pointwise limit.

The functions in Example 29.36.1 converge pointwise, but to a discontinuous function. However, they do not converge uniformly. We find $\|f - f_n\|_\infty = 1$ for all n. If they did, Theorem 29.38.1 tells us the limit would be continuous.

In the theorem below, we use $\{\mathbf{f}^n\}$ with superscripts to denote a sequence of functions in $\mathbb{R}^m$ so that we can use f_i^n for its i^{th} component.

Theorem 29.38.1. *Suppose* $\mathbf{f}^n \in \mathcal{C}_b(X; \mathbb{R}^m)$ *converges uniformly to a limit* $\mathbf{f}$. *Then* $\mathbf{f} \in \mathcal{C}_b(X; \mathbb{R}^m)$.

SKIPPED

Proof. We first show that $\mathbf{f}$ is a bounded function. This follows from the triangle inequality. Choose N with $\|\mathbf{f}^n - \mathbf{f}\|_\infty < 1$ for $n \geq N$. Then

$$\left\|\mathbf{f}\right\|_\infty \leq \left\|\mathbf{f} - \mathbf{f}^N\right\|_\infty + \left\|\mathbf{f}^N\right\|_\infty \leq 1 + \left\|\mathbf{f}^N\right\|_\infty.$$

It follows that each f_i is also bounded because

$$\left|f_i(\mathbf{x})\right| \leq 1 + \left\|\mathbf{f}^N(\mathbf{x})\right\|_{m,\infty} \leq 1 + \left\|\mathbf{f}^N\right\|_\infty$$

for all $\mathbf{x} \in X$ and $i = 1, \ldots, m$.

Proof continues ...

29.39 Uniform Limits of Continuous Functions II SKIPPED

Remainder of Proof. Next we show that each f_i is continuous. Suppose $\mathbf{x}_k \to \mathbf{x}$ and choose any $\varepsilon > 0$. Choose N with $\|\mathbf{f}^n - \mathbf{f}\|_\infty < \varepsilon$ for $n \geq N$. By continuity of $\mathbf{f}^N$, we may then choose $K \geq N$ with $\|\mathbf{f}^N(\mathbf{x}_k) - \mathbf{f}^N(\mathbf{x})\|_{m,\infty} < \varepsilon$ for $k \geq K$. Then for $k \geq K$ we have

$$\begin{aligned}
&\left|f_i(\mathbf{x}_k) - f_i(\mathbf{x})\right| \\
&\qquad \leq \left\|\mathbf{f}(\mathbf{x}_k) - \mathbf{f}(\mathbf{x})\right\|_{m,\infty} \\
&\qquad \leq \left\|\mathbf{f}(\mathbf{x}_k) - \mathbf{f}^N(\mathbf{x}_k)\right\|_{m,\infty} + \left\|\mathbf{f}^N(\mathbf{x}_k) - \mathbf{f}^N(\mathbf{x})\right\|_{m,\infty} + \left\|\mathbf{f}^N(\mathbf{x}) - \mathbf{f}(\mathbf{x})\right\|_{m,\infty} \\
&\qquad \leq \left\|\mathbf{f} - \mathbf{f}^N\right\|_\infty + \left\|\mathbf{f}^N(\mathbf{x}_k) - \mathbf{f}^N(\mathbf{x})\right\|_{m,\infty} + \left\|\mathbf{f} - \mathbf{f}^N\right\|_\infty \\
&\qquad < 3\varepsilon
\end{aligned}$$

Since ε was any positive number, this shows that each f_i and so $\mathbf{f}$ is continuous at $\mathbf{x}$. Since $\mathbf{x}$ was any point in X, $\mathbf{f}$ is continuous on all of X. ∎

Three-ε Argument. This is our first example of a 3-ε argument. One of its uses is to show continuity of the limit. Here the right-hand side is broken into three parts by twice adding and subtracting terms, and then using the triangle inequality to break it into three parts.

In this case, two of the parts are made smaller than ε by choosing N large, the third part is made small by using continuity of a particular $\mathbf{f}^N$ and making sure $\mathbf{x}_k$ and $\mathbf{x}$ are close enough together, which happens for $k \geq K$.

29.40 $\mathcal{C}_b(X;\mathbb{R}^m)$ is Uniformly Complete

The important fact about the uniform topology on $\mathcal{C}_b(X;\mathbb{R}^m)$ is that the uniform limit of continuous functions is continuous and that every uniformly Cauchy sequence has a uniform limit. Together, they show that $\mathcal{C}_b(X;\mathbb{R}^m)$ is a complete metric space in the uniform topology.

Theorem 29.40.1. *The space $\big(\mathcal{C}_b(X;\mathbb{R}^m), \|\cdot\|_\infty\big)$ is a complete metric space, and so a Banach space.*

SKIPPED

Proof. Suppose $\{\mathbf{f}^n\}$ is a Cauchy sequence in $\mathcal{C}_b(X;\mathbb{R}^m)$.

Step one is to find the limit. Let $\varepsilon > 0$. Then there is an N with $\|\mathbf{f}^k - \mathbf{f}^n\|_\infty < \varepsilon$ for $k, n \geq N$. It follows that for every $\mathbf{x} \in X$,

$$\left|f_i^k(\mathbf{x}) - f_i^n(\mathbf{x})\right| \leq \left\|\mathbf{f}^k - \mathbf{f}^n\right\|_{m,\infty} \leq \left\|\mathbf{f}^k - \mathbf{f}^n\right\|_\infty < \varepsilon$$

for all $k, n \geq N$ and $i = 1, \ldots, m$. In other words, each $\{f_i^n(\mathbf{x})\}$ is a Cauchy sequence and so has a limit $f_i(\mathbf{x})$. At this point we don't know whether $\mathbf{f}^n$ converges uniformly to $\mathbf{f}$. We only know that it converges pointwise to $\mathbf{f}$.

We finish the proof off by showing $\mathbf{f}^n$ converges uniformly to $\mathbf{f}$. At that point Theorem 29.38.1 shows that the limit is in $\mathcal{C}_b(X;\mathbb{R}^m)$.

Choose a new $\varepsilon > 0$ and N with $\|\mathbf{f}^k - \mathbf{f}^n\|_\infty < \varepsilon/2$ for $k, n \geq N$. Then for $\mathbf{x} \in X$,

$$\left|f_i^k(\mathbf{x}) - f_i^n(\mathbf{x})\right| \leq \left\|\mathbf{f}^k - \mathbf{f}^n\right\|_\infty < \varepsilon/2.$$

Taking the limit as $k \to \infty$ shows that $|f_i(\mathbf{x}) - f_i^n(\mathbf{x})| \leq \varepsilon/2$ for every $\mathbf{x} \in X$ and $i = 1, \ldots, m$. Then we can take the supremum over all $i = 1, \ldots, m$, and $\mathbf{x} \in X$ to find $\|\mathbf{f} - \mathbf{f}^n\|_\infty \leq \varepsilon/2 < \varepsilon$ for $n \geq N$. In other words, $\mathbf{f}^n \to \mathbf{f}$ uniformly, not just pointwise. ∎

29.41 Compact Sets: Introduction

One type of set of particular importance is the collection of compact sets. We will often be able to show that functions defined on compact sets have maxima and/or minima. This will ensure that standard economic problems such as utility maximization, expenditure minimization, cost minimization and profit maximization have solutions.

One of the odd things about compact sets is that we have not one, not two, but three definitions of compactness. When they all make sense, they are equivalent, but they don't all always apply. We start with a preliminary concept used in one of the definitions.

Cover. Let $S \subset X$. Any collection $\mathcal{U}$ of subsets of X obeying

$$S \subset \bigcup_{U \in \mathcal{U}} U$$

is called a *cover* of S. If $\mathcal{U}$ consists solely of open sets, we call $\mathcal{U}$ an *open cover* and if the collection $\mathcal{U}$ is finite, it is a *finite cover*.

If $\mathcal{U}$ is a cover of the set S, a *subcover* of the cover $\mathcal{U}$ is a subcollection $\mathcal{V} \subset \mathcal{U}$ that also covers S.

A Cover

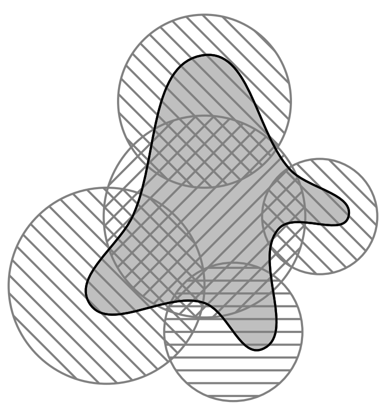

Figure 29.41.1: The gray set is covered by five balls.

29.42 Compact Sets: Definitions

There are three types of compactness. The first type applies to $\mathbb{R}^m$ with the usual topology. The second works in any metric space. The third applies to every topological space. Fortunately, whenever two or more apply, they all agree on which sets are compact.

Compact Set.

1. A set $S \subset \mathbb{R}^m$ is *closed and bounded compact* if it is closed and bounded.
2. A set S in a metric space (X, d) is *sequentially compact* if every sequence in S has a subsequence that converges to an element of S.
3. A set S is *Heine-Borel compact* if whenever $\{U_\alpha\}_{\alpha \in A}$ are open sets covering S, there is a finite subcover of S, $\{U_{\alpha_i}\}_{i=1}^N$.

These definitions are equivalent when two or more apply.

Consider the closed interval $S = [a, b] \subset \mathbb{R}$. The set S is compact. We know that the closed interval is closed, and it is bounded with bound $K = \max\{|a|, |b|\}$. Closed balls in $\mathbb{R}^m$ are also closed and bounded, hence compact.

Half-open intervals such as $(a, b]$ are bounded, but not closed (e.g., a is a limit point of the set, but not part of the set). The interval is not compact because $x_n = a + 1/n$ is in the interval and $x_n \to a$, but a is not in the interval.

29.43 Sequential Compactness in $\mathbb{R}^m$

Theorem 29.43.1. *Suppose $S \subset \mathbb{R}^m$ is sequentially compact in the usual topology. Then it is closed and bounded.*

Proof. Let $\{x_n\}$ be a sequence in S with $x_n \to x$. Since S is sequentially compact, $x \in S$, showing that S is closed.

Now suppose S is not bounded. Then for every integer $n > 0$ there is a $x_n \in S$ with $\|x_n\| > n$ (Lemma 29.16.1). By sequential compactness this has a convergent subsequence x_{n_k} with $x_{n_k} \to x \in S$. By Theorem 13.21.1, $\|x_{n_k}\| \to \|x\| < \infty$. But by construction, $\|x_{n_k}\| \to \infty$. This **contradiction shows** that S **must be bounded**. ∎

29.44 The Bolzano-Weierstrass Theorem

Because we have done things in a different order than Simon and Blume, we can use a different proof for the Bolzano-Weierstrass Theorem.[8]

Bolzano-Weierstrass Theorem. *Any box* $B = \prod_{i=1}^{m}[a_i, b_i] \subset \mathbb{R}^m$ *is sequentially compact.*

Proof. Let $\{\mathbf{x}^n\}$ be a sequence in B and let x_i^n be the i^{th} coordinate of $\mathbf{x}^n$. By Theorem 29.25.1, x_1^n has a monotone subsequence $x_1^{n_k^1}$. Similarly, $x_2^{n_k^1}$ has a monotone subsequence $x_2^{n_k^2}$. Continue taking subsequences until we run out of coordinates (there are only m of them), obtaining $\mathbf{y}^k = \mathbf{x}^{n_k^m}$ which is monotone in every coordinate. It is also bounded in every coordinate, and so converges in every coordinate by Theorem 29.24.1. Finally, Theorem 12.13.1 shows this subsequence converges in the ℓ_2 norm. Since B is closed, $\lim_k \mathbf{y}^k \in B$, showing that B is sequentially compact. ∎

[8] Like Cauchy, the Bohemian mathematician, philosopher, theologian, and priest Bernard Bolzano (1781–1848), was one of the precursors to the 19^{th} century project of putting mathematical analysis on a sound foundation. Unfortunately, some of his key mathematical results were not published until 1930(!) including his work on uniform continuity. Bolzano also found the first example of a function that was continuous but nowhere differentiable, but it wasn't published until 1922. Oddly enough, there were 3 other unpublished examples of such a function prior to Weierstrass's publication of one in 1872.

Karl Weierstrass (1815–1897) later built on Bolzano's published work when revamping the foundations of analysis in the mid to late 19^{th} century.

The term "compact" was introduced by Fréchet in his 1906 Ph.D. dissertation, where he generalized the Bolzano-Weierstrass Theorem (and sequential compactness) to spaces of functions.

29.45 Consequences of the Bolzano-Weierstrass Theorem

Lemma 29.45.1. *If S is sequentially compact and T is a closed subset of S, then T is sequentially compact.*

Proof. Let $\{\mathbf{x}_n\}$ be a sequence in $T \subset S$. Since S is sequentially compact, we can find a subsequence $\mathbf{x}_{n_k}$ that converges to a point in S. As T is closed, and each $\mathbf{x}_{n_k} \in T$, $\lim_k \mathbf{x}_{n_k} \in T$, showing that T is sequentially compact. ∎

We can now show that closed and bounded sets in $\mathbb{R}^m$ are sequentially compact. Since we already showed in Theorem 29.43.1 that sequentially compact sets in $\mathbb{R}^m$ are closed and bounded, this means that the first two definitions of compactness are equivalent on $\mathbb{R}^m$.

Corollary 29.45.2. *A set S is a closed and bounded set in $\mathbb{R}^m$, if and only if it is sequentially compact.*

Proof. **Only If:** Since S is bounded, it is contained in $[-K, K]^m$ for some K. Now $[-K, K]^m$ is sequentially compact by the Bolzano-Weierstrass Theorem, and S is a closed subset of $[-K, K]^m$, so S is also sequentially compact.

The converse was shown in Theorem 29.43.1. ∎

29.46 Heine-Borel Compact Sets and Subsets

It's easy to show that a closed subset of a Heine-Borel compact set is Heine-Borel compact.[9][10]

Theorem 29.46.1. *Suppose* S *is Heine-Borel compact and* $T \subset S$ *is closed. Then* T *is Heine-Borel compact*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of T. Since T^c is open, $\{T^c\} \cup \{U_\alpha\}_{\alpha \in A}$ is an open cover of S. It has a finite subcover, $\{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ and possibly T^c. It follows that $\{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ is a finite subcover of T (T^c can never cover any part of T). Thus T is Heine-Borel compact. ∎

[9] The German mathematician Eduard Heine (1821–1881) worked mostly in real analysis. Besides compactness, he's known for his work on uniform continuity and various special functions.

[10] The French mathematician Émile Borel (1871–1956) did major work in measure theory and probability such as the Borel-Cantelli Lemma and Borel distribution. He should not be confused with the Swiss mathematician Armand Borel (1923–2004). It remains unclear whether they were relatives.

29.47 Heine-Borel and Sequential Compactness

In metric spaces, Heine-Borel compactness and sequential compactness are the same. One advantage of the Heine-Borel definition is that it applies in more spaces, in spaces that don't have a metric topology.

Theorem 29.47.1. *A set S in a metric space* (X, d) *is Heine-Borel compact if and only if it is sequentially compact.*

Proof. Only if case ($\Rightarrow$): Here S is Heine-Borel compact. **If** S **is not sequentially compact**, there must be a sequence $\{\mathbf{x}_n\}$ that has no convergent subsequence. It follows that no point in $\{\mathbf{x}_n\}$ can be repeated infinitely often. Otherwise we could take a subsequence that is only that point, and so converges.

For any $\mathbf{x} \in S$, if for every $\varepsilon > 0$, $B_\varepsilon(\mathbf{x})$ contains at least one of the $\mathbf{x}_n$, we can construct a convergent subsequence by taking $\varepsilon = 1/k$ and choosing $\mathbf{x}_{n_k} \in B_{1/k}(\mathbf{x})$. Then $\lim_k \mathbf{x}_{n_k} = \mathbf{x}$. As this is impossible, there is always an ε_x with $B_{\varepsilon_x}(\mathbf{x})$ containing no $\mathbf{x}_n$ except possibly $\mathbf{x}$.

The $B_{\varepsilon_x}(\mathbf{x})$ are an open cover of S, so they have a finite subcover. Now the union of finitely many of these balls contains at most finitely many terms of the sequence $\{\mathbf{x}_n\}$—it's important here that no point is infinitely repeated. It follows that the open subcover doesn't cover S! **This contradiction** implies that S is sequentially compact, that every sequence has a convergent subsequence.

If case ($\Leftarrow$): Omitted, it takes us too far afield. ∎

At this point we know that the different types of compactness are the same whenever they all make sense. This means that we can use the term *compact* without worrying which type of compactness we mean. They are all the same.

30. Calculus of Several Variables II

30.1 Continuous Image of a Compact Set is Compact

One important result is that a continuous image of a compact set is also compact.

Theorem 30.1.1. *Let $f\colon X \to Y$ where X and Y are topological spaces. If f is continuous and $S \subset X$ is compact, $f(S)$ is also compact.*

Proof. Let $\{V_\alpha\}_{\alpha\in A}$ be an open cover of $f(S)$. Let $U_\alpha = f^{-1}(V_\alpha)$. Then the collection $\{U_\alpha\}$ is an open cover of S. It has a finite subcover $U_{\alpha_1}, \ldots, U_{\alpha_k}$. It follows that $V_{\alpha_1}, \ldots, V_{\alpha_k}$ is a finite subcover of $f(S)$. Thus $f(S)$ is compact. ∎

30.2 Weierstrass's Theorem

For our purposes, one of the most useful results from topology is Weierstrass's Theorem.[1] It is the key to showing that many economic problems, such as the consumer's utility maximization problem, have solutions

Weierstrass's Theorem. *Let $S \subset \mathbb{R}^m$ be compact and $f\colon S \to \mathbb{R}$ be continuous. Then there are $\mathbf{x}_* \in S$ and $\mathbf{x}^* \in S$ with $f(\mathbf{x}_*) \le f(\mathbf{x}) \le f(\mathbf{x}^*)$ for all $\mathbf{x} \in S$. In other words, f attains both a maximum and minimum on S.*

Proof. Now f is continuous and S compact, so $f(S)$ is compact by Theorem 30.1.1. Since $f(S) \subset \mathbb{R}$, it is closed and bounded. Thus $\sup f(S) \in f(S)$ and $\inf f(S) \in f(S)$. Taking $x^* \in S$ with $f(x^*) = \sup f(S)$ and $x_* \in S$ with $f(x_*) = \inf f(S)$ completes the proof. ∎

▶ **Example 30.2.1: Cases where Weierstrass Doesn't Apply.** If the set is not compact, a maximum may not exist. For example, if $S = [0, 1)$ and $f(x) = x$, there is no maximum. It would be at 1, if 1 were in S, but it isn't.

If the function is not continuous, a maximum may not exist. Let $S = [0, 1]$ and define $f(x) = x$ for $x < 1$ and $f(1) = 0$. Once again, the problem occurs at $x = 1$, but here the problem is that the function jumps downward. ◀

[1] This theorem is also known as Weierstrass's Extreme Value Theorem to distinguish it from other Weierstrass Theorems. The first known version was proved by Bernard Bolzano in the 1830's, but not published until 1930.

30.3 Utility Maximization

We consider the problem of maximizing a continuous utility function $u\colon \mathbb{R}^m \to \mathbb{R}$ subject to the budget constraint $\mathbf{p}\cdot\mathbf{x} \le m$ and the non-negativity constraints $x_1, \ldots, x_m \ge 0$ where $\mathbf{p} \gg \mathbf{0}$ and $m \ge 0$. In other words, we ask whether the standard consumer's problem of microeconomics has a solution. Fortunately for us, it does.

Theorem 30.3.1. *Let $u\colon \mathbb{R}^m \to \mathbb{R}$ be a continuous utility function. Suppose $\mathbf{p} \gg \mathbf{0}$ and $m \ge 0$. Then the consumer's utility maximization problem, maximizing utility u over the budget set $B(\mathbf{p}, m) = \{\mathbf{x} \in \mathbb{R}^m_+ : \mathbf{p}\cdot\mathbf{x} \le m\}$ has a solution.*

Proof. Let $B(\mathbf{p}, m)$ denote the budget set. Now $\mathbf{0} \in B(\mathbf{p}, m)$, so the budget set is non-empty. In Example 13.35.1, we found that such budget sets are closed, while Example 29.17.1 showed they are bounded. This means that the budget set is compact.

Weierstrass's Theorem now tells us that the continuous function u can be maximized on $B(\mathbf{p}, m)$. ∎

One economic problem where the Weierstrass Theorem may fail is profit maximization.

► Example 30.3.2: Unmaximizable Profit. Suppose a firm has production function $f(x) = 2x$ and faces output price $p = 1$ and input price $w = 1$. Profit is $pf(x) - wx = x$. Here profit increases without bound as input x increases. Profit cannot be maximized here. ◄

30.4 Utility Maximization can be Impossible

The proof that utility can be maximized relies on a continuous utility function (allowing use of Weierstrass's Theorem) and strictly positive prices (bounding the budget set). If either condition fails—either some price is zero or preferences are not continuous—the consumer's utility maximization problem may not have a solution. The first case is considered in the next example, which sets one of the prices to zero, and finds there is no solution.

▶ **Example 30.4.1: No Utility Max with Zero Price.** In $\mathbb{R}^2_+$, let utility be $u(x_1, x_2) = x_1 x_2$, $m = 1$ and $\mathbf{p} = (1, 0)$. We encountered this unbounded budget set in Figure 29.19.2. Here utility is continuous, but one of the prices is zero. The point $(1, n)$ is in the budget set for any n. It yields utility $u(1, n) = n$. There is no utility maximum as taking n sufficiently large would be better than any would-be maximum. ◀

There are times when utility maximization is possible even with a zero price. If we replace the utility function in Example 30.4.1 with $v(x_1, x_2) = x_1 - (x_2 - 10)^2$, there is a utility maximum at $\mathbf{x} = (1, 10)$.

30.5 Utility Maximization with Discontinuous Utility

The second case, discontinuous utility, is the subject of the following example. Again, the consumer's utility maximization problem cannot be solved.

▶ **Example 30.5.1: No Utility Max with Discontinuous Utility.** Again, define utility on $\mathbb{R}^2_+$, this time with

$$u(x_1, x_2) = \begin{cases} x_1 & \text{when } x_2 > 0 \\ 0 & \text{when } x_2 = 0. \end{cases}$$

This utility function is not continuous. Consider the case $m = 1$ and $\mathbf{p} = (1, 1)$. Since $x_1 \leq 1$, we know that the maximum utility can be no more than 1. However, attaining utility 1 requires $x_1 = 1$. The budget constraint becomes $1 + x_2 \leq 1$, implying $x_2 = 0$. This means utility is actually zero.

If we keep $x_2 > 0$, $x_1 = 1 - x_2$, so $u(x_1, x_2) = 1 - x_2 < 1$. By taking x_2 very small, we can get as close to one *util* as we like, but cannot actually attain it. There is no maximum because any utility level less than one can be beaten, while utility one cannot be attained. ◀

The failure of compactness of $B(\mathbf{p}, m)$ or of continuity of preference does not necessarily mean that there will be no solution. There are times when the consumer's utility maximization problem has a solution even though utility is discontinuous and the budget set is not compact.

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