15. Implicit Functions and Their Derivatives

When y is written as a function of $(x_1, \ldots, x_m)$,

$$y = f(x_1, \ldots, x_m)$$

we say that y is an *explicit function of* $(x_1, \ldots, x_m)$.

Things are different when y and $(x_1, \ldots, x_m)$ are combined in a single function so that

$$f(x_1, \ldots, x_m, y) = 0. \tag{15.0.1}$$

If the $x_1, \ldots, x_m$ determine y in equation (15.0.1), we say that y is an *implicit function of* $(x_1, \ldots, x_m)$

With luck, we will be able to solve for y in terms of $(x_1, \ldots, x_m)$. But that is not always possible. For example, the quintic equation $y^5 - 5xy + 4x^2 = 0$ does not have an explicit solution, although we can say that $(x, y) = (1, 1)$ is a solution, as is $(1/4, 1)$, suggesting that $y(1) = 1$ and $y(1/4) = 1$. It's also clear that $y(0) = 0$. There are hints of a function here, but we can't solve for it.

When the equation implicitly defines y in terms of x, but we cannot write an expression for $y(x)$, we might still be able to determine the derivatives. The Implicit Function Theorem gives conditions for finding local functions for y and their derivatives.

15.1 Is it an Implicit Function?

One issue with equation (15.0.1) is that it is difficult to determine whether there even is an implicit function.

▶ **Example 15.1.1: No Implicit Function for a Circle.** Consider the equation $x^2+y^2 = 25$. Does this implicitly define $y(x)$? In this case we can use the quadratic formula to solve for y, obtaining

$$y(x) = \pm\sqrt{25-x^2}.$$

There is a problem here. This is not a function!

For values of $x \in (-5, +5)$, there are two values of $y(x)$, not one. Only at $x = \pm 5$ do we have a function. Everywhere else there are two values of y for every x. This is illustrated in Figure 15.2.2 where there are two values of y corresponding to $x = 3$.

One way to work around this is to lower the bar, to give up the search for a global function and focus on a locally defined implicit function. We look for a function that solves the equation in a neighborhood of a point (x_0, y_0). We can use one function near $(3, 4)$ and another near $(3, -4)$.

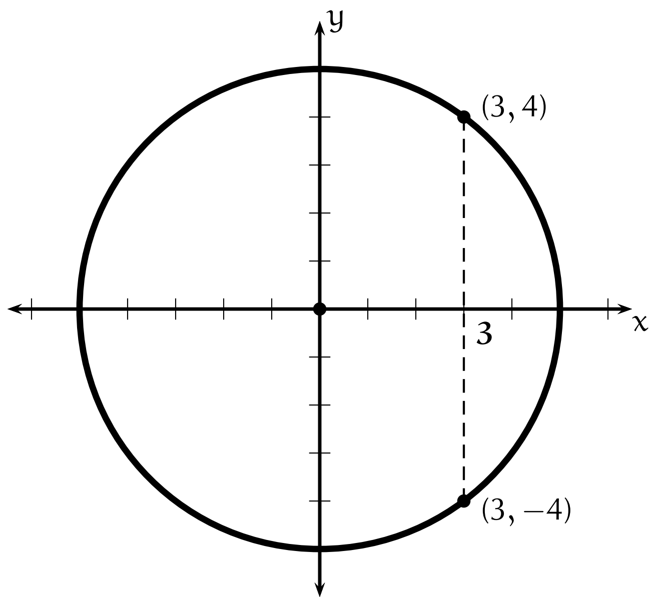

Figure 15.1.2: The circle is the graph of $x^2 + y^2 = 25$, which tries to implicitly define y as a function of x. As you can see, there are two solutions $y(x)$ for most values of x. This is illustrated at $x = 3$.

◀

15.2 Picking an Implicit Function

► **Example 15.2.1: Local Implicit Functions on a Circle.** Here $y = +(25 - x^2)^{1/2}$ is implicitly defined by the equation $x^2 + y^2 = 25$ and includes the starting point $(x_0, y_0) = (3, 4)$. It can be defined on open sets as large as $(-5, 5)$. Similarly, if $(x_0, y_0) = (3, -4)$, the function $y = -(25 - x^2)^{1/2}$ works for $x \in (-5, 5)$.

For the function in question, that all works fine at most points on the circle. However, a problem occurs at both $(5, 0)$ and $(-5, 0)$. Neither point allows us to define y on an open interval containing $x = \pm 5$. The points $x = \pm 5$ cannot be in the interior of the domain of y.

This is connected with the fact that the graph becomes vertical at those two points.

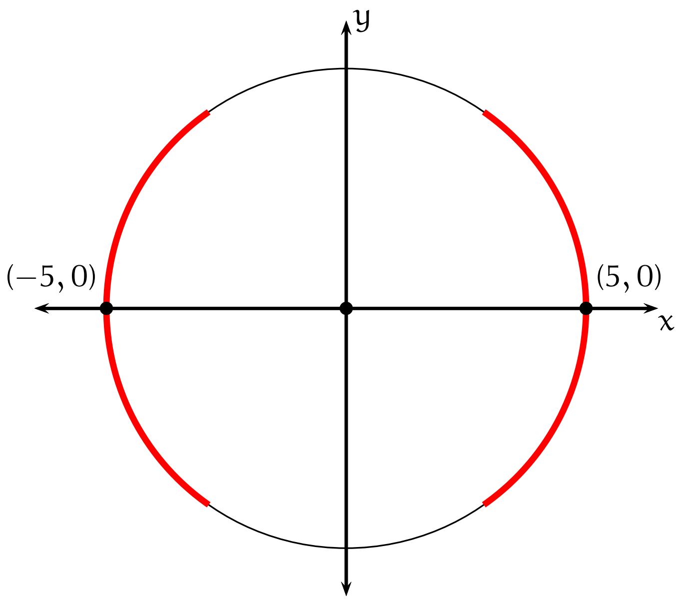

Figure 15.2.2: The circle is the graph of $x^2 + y^2 = 25$, which tries to implicitly define y as a function of x. The points $(5, 0)$ and $(-5, 0)$ pose particular problems as we are unable to write y as a function of x on a neighborhood of $x = \pm 5$ due to the verticality of the graph of y at $x = \pm 5$.

◄

15.3 The Implicit Function Theorem for $\mathbb{R}^2$

The key result is the Implicit Function Theorem. Here is a version for $\mathbb{R}^2$. The condition $(\partial G/\partial y)(x_0, y_0) \neq 0$ rules out vertical graphs at (x_0, y_0).

Implicit Function Theorem for $\mathbb{R}^2$**.** *Let* $G(x, y)$ *be a* $\mathcal{C}^1$ *function on a neighborhood of* $(x_0, y_0) \in \mathbb{R}^2$*. Suppose that* $G(x_0, y_0) = c$*. If*

$$\frac{\partial G}{\partial y}(x_0, y_0) \neq 0,$$

there exists a $\mathcal{C}^1$ *function* $y(x)$ *defined on an interval* I *containing* x_0 *such that:*

(a) $G\big(x, y(x)\big) = c$ *for all* $x \in I$*,*
(b) $y(x_0) = y_0$*, and*
(c) *The function* y *obeys*

$$y'(x_0) = -\frac{\dfrac{\partial G}{\partial x}(x_0, y_0)}{\dfrac{\partial G}{\partial y}(x_0, y_0)}.$$

► Example 15.3.1: The Theorem and the Circle. How does this apply to the circle $x^2 + y^2 = 25$ we studied in Example 15.2.1? Here we set $G(x, y) = x^2 + y^2$ and $c = 25$. Let's try $(x_0, y_0) = (3, -4)$ and see what happens.

Here $(\partial G/\partial y)(3, -4) = -8 \neq 0$, so we can apply the Implicit Function Theorem to find $y(x)$ solving $x^2 + [y(x)]^2 = 25$ with $y(3) = -4$ and $y'(3) = -2(3)/2(-4) = 3/4$. Compare to the solution y_1 given by $y_1(x) = -(25 - x^2)^{1/2}$. Then

$$y_1'(x) = -(1/2)(25 - x^2)^{-1/2}(2x) = \frac{x}{\sqrt{25 - x^2}}.$$

Then $y_1'(3) = 3/4$, exactly as with y. ◄

The Implicit Function Theorem will be useful for writing one set of economic variables as a function of other variable. For instance, suppose we solve the consumer's problem for prices $\mathbf{p}$ and income m. Can we write the demands for x and y as functions of prices and income? Once we characterize the solution via first order and second order equations, we will be able to use the Implicit Function Theorem to find whether we have proper demand functions.

15.4 Implicit Function Theorem: Sketch of Proof

Although we won't dot every i and cross every t, we will cover the basic idea behind the Implicit Function Theorem.

Sketch of Proof. By replacing G by $G(x, y) - c$, we may assume $G(x_0, y_0) = 0$. Also $(\partial G/\partial y)(x_0, y_0) \neq 0$ by hypothesis. We may also assume $(\partial G/\partial y)(x_0, y_0) > 0$ (otherwise, replace G by $-G$).

Now $G \in \mathcal{C}^1$, so $\frac{\partial G}{\partial y}$ is continuous. It follows that there is a small square

$$S = \{(x, y) : x_0 - \varepsilon \leq x \leq x_0 + \varepsilon, y_0 - \varepsilon \leq y \leq y_0 + \varepsilon\}$$

with $\partial G/\partial y > 0$ on S.

Keeping in mind that $\partial G/\partial y$ is bounded away from zero on the compact set S and $G(x_0, y_0) = 0$, we may choose $\varepsilon_0 > 0$ small enough that $G(x, y_0 - \varepsilon) < 0 < G(x, y_0 + \varepsilon)$ whenever $|x - x_0| \leq \varepsilon_0$. Now define the rectangle R by

$$R = \{(x, y) : |x - x_0| \leq \varepsilon_0, |y - y_0| \leq \varepsilon\}.$$

On R, $\partial G/\partial y > 0$, and for $(x, y_0 \pm \varepsilon) \in R$, $G(x, y_0 - \varepsilon) < 0$ and $G(x, y_0 + \varepsilon) > 0$.

Set $I = [x_0 - \varepsilon_0, x_0 + \varepsilon_0]$. Now G is continuous on R and takes opposite signs at the top and bottom of R. For each $x \in I$ the Intermediate Value Theorem yields a $y(x)$ with $G\big(x, y(x)\big) = 0$. Moreover, because G is increasing in y on R, there is only one such point $y(x)$ for each $x \in I$.

Now suppose $x_n \to x$ with $x_n \in I$. Because I is a closed interval, $x \in I$. Now consider $y_n = y(x_n)$. Because R is a compact interval, there is a subsequence (x_{n_k}, y_{n_k}) that converges to a point of R. Since $x_n \to x$, $x_{n_k} \to x$, and the only question is what is $y = \lim_k y_{n_k}$. We know $G(x_{n_k}, y_{n_k}) = 0$ and G is continuous, so $G(x, y) = 0$. Since this equation has a unique solution in R, $y = y(x)$, $y_{n_k} \to y(x)$. This means that all convergent subsequences of $y(x_n)$ have limit $y(x)$, so $\lim_n y(x_n) = y(x)$, showing that y is continuous.

The only thing left to do is show that y is a $\mathcal{C}^1$ function on I^0. As this is more technical than illuminating, we will skip that step. ∎

15.5 One-dimensional Differentiable Manifolds

Regular Points and Curves. A point (x_0, y_0) is a *regular point* of a $\mathcal{C}^1$ function $G\colon \mathbb{R}^2 \to \mathbb{R}$ if either

$$\frac{\partial G}{\partial x}(x_0, y_0) \neq 0 \quad \text{or} \quad \frac{\partial G}{\partial y}(x_0, y_0) \neq 0.$$

If every point on $C = \{(x, y) : G(x, y) = c\}$ is a regular point, we say that C is a *regular curve* or a *one-dimensional differentiable manifold*.

The following theorem characterizes the regular points of a curve. It follows immediately from the Implicit Function Theorem for $\mathbb{R}^2$.

Theorem 15.5.1. *Let $G\colon \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^1$ function. If (x_0, y_0) is a regular point on the curve $C = \{(x, y) : G(x, y) = c\}$, Then either*

1. $(\partial G/\partial y)(x_0, y_0) \neq 0$ *and there is a $\mathcal{C}^1$ function $y(x)$ with $G\big(x, y(x)\big) = c$ on some neighborhood of (x_0, y_0), or*
2. $(\partial G/\partial x)(x_0, y_0) \neq 0$ *and there is a $\mathcal{C}^2$ function $x(y)$ with $G\big(x(y), y\big) = c$ on some neighborhood of (x_0, y_0).*

In the former case,

$$y'(x_0) = -\frac{\dfrac{\partial G}{\partial x}(x_0, y_0)}{\dfrac{\partial G}{\partial y}(x_0, y_0)}$$

and in the latter

$$x'(y_0) = -\frac{\dfrac{\partial G}{\partial y}(x_0, y_0)}{\dfrac{\partial G}{\partial x}(x_0, y_0)}.$$

When the curve defined by G is regular, we can strengthen this as follows.

Corollary 15.5.2. *Let $G\colon \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^1$ function. It the curve $C = \{(x, y) : G(x, y) = c\}$ is regular, at every point (x_0, y_0) on the curve C, we can parameterize the curve by a $\mathcal{C}^1$ curve defined on an open set containing (x_0, y_0): either as $\big(x, y(x)\big)$ or $\big(x(y), y\big)$.*

The manifold C is considered one-dimensional since it can be locally described by a single parameter. It is differentiable because there are invertible differentiable functions that describe it locally.

15.6 Tangent Spaces

What if we have a curve inside our manifold? Can we say anything about its tangent vector?

One way to define tangent vectors for a manifold M is to define them as the tangent vectors of all the curves in M. To that end, let $M = \{(x,y) \in \mathbb{R}^2 : G(x,y) = c\}$ be a regular manifold and $\mathbf{x}(t) = \big(x(t), y(t)\big)$ be a $\mathcal{C}^1$ curve in M. It obeys $G\big(x(t), y(t)\big) = c$ for all $t \in I$. Then we can consider $\boldsymbol{v} = \mathbf{x}'(t_0)$ to be a tangent vector to M at $\mathbf{x}(t_0) = \big(x(t_0), y(t_0)\big)$.

Now $G\big(x(t), y(t)\big) = c$ for all t. We apply the Chain Rule at t_0 to find

$$DG(x_0, y_0)\big(x'(t_0), y'(t_0)\big) = \big[DG(x_0, y_0)\big]\boldsymbol{v} = 0.$$

The tangent vector $\boldsymbol{v}$ is in $\ker DG(x_0, y_0)$. In fact, any element of the kernel can be represented this way by appropriate choice of $\mathbf{x}(t)$.[1] We call $T_{(x_0,y_0)}M = \ker DG(x_0, y_0)$ the *tangent space* of M at (x_0, y_0).

Alternatively, we relate the tangent space $T_{(x_0,y_0)}M$ to the gradient vector $\nabla G(x_0, y_0)$:

$$\nabla G(x_0, y_0) \cdot \boldsymbol{v} = 0.$$

The gradient gives the direction of fastest increase of G. The tangent space $T_{(x_0,y_0)}M$ is the set of all perpendicular to it (definition 2).

[1] This will be easier to see once we introduce coordinate charts.

15.7 Homeomorphisms

A homeomorphism is a continuous map with a continuous inverse. More precisely,

Homeomorphism. Let X and Y be topological spaces. A bijective map $\varphi\colon X \to Y$ is a *homeomorphism* between X and Y if both φ and φ^{-1} are continuous. Where there is a homeomorphism between X and Y, we say X and Y are *homeomorphic*.

Homeomorphisms are one of the fundamental concepts of topology. In a sense, they are the defining concept. Homeomorphisms preserve topological properties such as openness, closedness, connectedness, compactness, and a number of others I haven't mentioned. If a property is not preserved by a homeomorphism, it isn't considered a topological property.

Homeomorphisms can be composed. If $f\colon X \to Y$ is a homeomorphism and $g\colon Y \to Z$ is a homeomorphism, then $g \circ f\colon X \to Z$ is also a homeomorphism. Together with the fact that the identity map $i(x) = x$ is a homeomorphism from X to X, it shows that being homeomorphic is an equivalence relation.

That is, being homeomorphic, is reflexive (X is homeomorphic to itself by i), symmetric (by definition, X is homeomorphic to Y if and only if Y is homeomorphic to X), and transitive (if X is homeomorphic to Y and Y homeomorphic to Z, then X is homeomorphic to Z by composition).

Homeomorphisms have the important property that image of an open set is open.

Theorem 15.7.1. *Suppose* $\varphi\colon X \to Y$ *is a homeomorphism and* $U \subset X$ *is a open set. Then* $\varphi(U)$ *is also open.*

Proof. By hypothesis, $\psi = \varphi^{-1}$ is continuous. Now $\varphi(U) = \psi^{-1}(U)$, which is open as the continuous inverse image of an open set, U. ∎

15.8 Examples of Homoeomorphism

Here are three homeomorphism. It is fairly easy to verify that all are bijective continuous mappings with continuous inverses.

▶ **Example 15.8.1: Stretching and Compressing** $\mathbb{R}^m$. A simple example of a homeomorphism between $\mathbb{R}^m$ and $\mathbb{R}^m$ is $f(\mathbf{x}) = 50\mathbf{x}$. This function stretches $\mathbb{R}^m$. The images of two points are always 50 as far apart as the points themselves.

Its inverse is a contraction, compressing distances by a factor of 50. The inverse is defined by $f^{-1}(\mathbf{x}) = (1/50)\mathbf{x}$. Then

$$\|f^{-1}(\mathbf{x}) - f^{-1}(\mathbf{y})\|_2 = \frac{1}{50}\|\mathbf{x} - \mathbf{y}\|_2$$

Both f and f^{-1} map $\mathbb{R}^m$ to $\mathbb{R}^m$. ◀

▶ **Example 15.8.2: Graph of a Parabola.** Another homeomorphism maps $\mathbb{R}$ onto the parabola $S = \{(x, x^2) : x \in \mathbb{R}\}$, where the parabola S is given its relative topology in $\mathbb{R}^2$. The map $f\colon \mathbb{R} \to S$ is defined by $f(x) = (x, x^2)$ and its inverse $f^{-1}\colon S \to \mathbb{R}$ is given by projection onto the first coordinate: $f^{-1}(x, x^2) = x$. ◀

▶ **Example 15.8.3: Homeomorphism between a Half-Circle and the Real Line.** This homeomorphism maps the interval $(-1, +1)$ onto the real line. Here

$$f(t) = \frac{t}{\sqrt{1 - t^2}} \quad \text{and} \quad f^{-1}(x) = \frac{x}{\sqrt{1 + x^2}}.$$

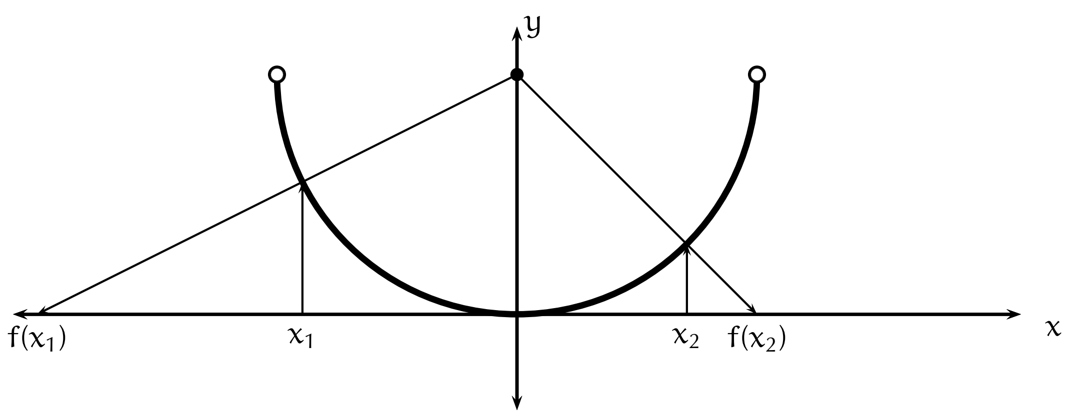

Figure 15.8.4: The diagram shows how the mapping f works. We start with a point $x \in (-1, 1)$, map it straight up to the semicircle of radius one centered at $(0, 1)$, then project the line though $(0, 1)$ and the resulting point back to the x-axis to obtain $f(x)$. Two samples are shown, from $x_1 = -2/\sqrt{5}$ and $x_2 = +5/\sqrt{2}$.

◀

15.9 Manifolds and Dimension

Manifolds can be defined very simply.[2]

Manifolds. A metric space M is a *manifold* if for every $x \in M$, there is a neighborhood U of x and some integer $m \geq 0$ such that U is homeomorphic to an open subset of $\mathbb{R}^m$.

This type of manifold is sometimes called a *topological manifold*. While differentiable manifolds are topological manifolds, not all topological manifolds are differentiable manifolds.

So far we have thought of dimension in algebraic terms, using vector spaces and bases. Dimension also has topological aspects. There is an important theorem in topology that relates topological dimension and homeomorphism. We state it without proof.

Invariance of Domain. *Let U be an open set in $\mathbb{R}^m$ and $f\colon U \to \mathbb{R}^m$ be one-to-one. Then $V = f(U)$ is open in $\mathbb{R}^m$ and f is a homeomorphism between U and V.*

As a result, it's not possible to have a homeomorphism between $\mathbb{R}^k$ and $\mathbb{R}^m$ for $k \neq m$. Although we don't show it, it's not even possible to have homeomorphisms between open sets in $\mathbb{R}^k$ and $\mathbb{R}^m$ unless $k = m$.

Lemma 15.9.1. *If $f\colon \mathbb{R}^k \to \mathbb{R}^m$ is a homeomorphism, then $k = m$.*

Proof. **If** $k > m$, we can regard $\mathbb{R}^m$ as the first k coordinates in $\mathbb{R}^k$. That is, define $\hat{f}(\mathbf{x}) = \big(f_1(\mathbf{x}), \dots, f_m(\mathbf{x}), 0, \dots, 0\big)$. Then $\hat{f}\colon \mathbb{R}^k \to \mathbb{R}^k$. Now $\hat{f}$ is one-to-one, so $\hat{f}(\mathbb{R}^m)$ must be open by Invariance of Domain. But it isn't because the last $k - m$ coordinates are zero. **This contradiction** shows $k \leq m$.

If $k > m$, consider its inverse and apply the above argument, which again **leads to contradiction**.

It follows that $k = m$. ∎

Now both topology and linear algebra now agree that $\mathbb{R}^m$ is m-dimensional.

[2] See page 1-1 of Michael Spivak, *A Comprehensive Introduction to Differential Geometry*, vol. I, 1970. Also see page 1 of Morris Hirsch, *Differential Topology*, 1976.

15.10 Circles are Manifolds

One example of a one-dimensional manifold is any circle in $\mathbb{R}^2$. We will show this for the circle defined by $x^2 + y^2 = 25$, but the methods used apply to any circle. In fact, they can also be applied to spheres in $\mathbb{R}^m$.

► **Example 15.10.1: A Circle in $\mathbb{R}^2$.** We continue to study the circle in Example 15.2.1, Let $C = \{(x, y) : x^2 + y^2 = 25\}$. We will use four functions to describe C as a manifold.

$$\begin{aligned}
g_1(x) &= \left(x, \sqrt{25 - x^2}\right) && \text{for } x \in (-5, 5)\\
g_2(x) &= \left(x, -\sqrt{25 - x^2}\right) && \text{for } x \in (-5, 5)\\
g_3(y) &= \left(\sqrt{25 - y^2}, y\right) && \text{for } y \in (-5, 5)\\
g_4(y) &= \left(-\sqrt{25 - y^2}, y\right) && \text{for } y \in (-5, 5)
\end{aligned}$$

The functions g_1 and g_2 describe the top and bottom halves of the circle, respectively. while g_3 and g_4 describe the left and right halves of the circle, resectively. The functions can be defined on the closures of the above intervals, but the functions cannot be $\mathcal{C}^1$ there.

We can use the functions g_i to map pieces of the circle C to and from intervals in $\mathbb{R}$. Here we identify the interval $(-5, +5)$ with $\{(x, 0) : -5 < x < 5\}$ and $\{(0, y) : -5 < y < 5\}$ when the latter two are given their relative topology in $\mathbb{R}^2$.

Set $U_1 = \{(x, y) : y > 0\}$ and define the projection $\varphi_1 \colon U_1 \cap C \to (-5, 5)$ by $\varphi_(x, y) = x$. Then $g_1 \circ \varphi \colon U_1 \cap C \to U_1 \cap C$ and is defined by $g_1\big(\varphi_1(x, y)\big) = y$. Here both φ_1 and its inverse g_1 are continuous, one-to-one, and onto. Therefore φ_1 is a homeomorphisms between $U_1 \cap C$ and the interval $(-5, 5)$. ◄

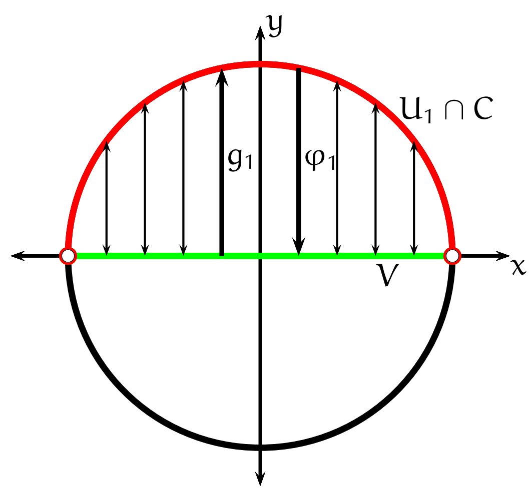

Figure 15.10.2: The vertical lines illustrate the bijection between $U_1 \cap C$ (in red) and $V = \{(x, 0) : |x| < 5\}$ (green) created by projection φ_1 onto the x-axis and g_1, mapping back to the circle. Two examples are highlighted. One showing φ_1 mapping down to the x-axis, the other showing g_1 mapping up to the circle.

Here $V = \{(x, 0) : x \in (-5, 5)\}$ has the relative topology, allowing us to identify it with $(-5, +5)$ via the homeomorphism $\psi(x, 0) = x$.

15.11 Coordinate Charts

10/20/20

Homework: Problems 22 and 36 from Chapter 15, problems 2 and 3 from Chapter 16 and problem 7 from Chapter 30 are due on Tuesday, October 27.

So far, we have only examined one-dimensional manifolds using the Implicit Function Theorem. It will be helpful to upgrade our definition before going further. We start by establishing coordinate charts, which allow us to describe open sets in a manifold M in terms of a coordinate system in some $\mathbb{R}^m$.

The term *chart* refers to inverses of functions like those in Example 15.10.1.

Chart. A *chart* or *coordinate system* on a manifold M is a pair (U, φ) where U is a open subset of M and φ is a continuous homeomorphism from U to an open subset of some Euclidean space $\mathbb{R}^m$.

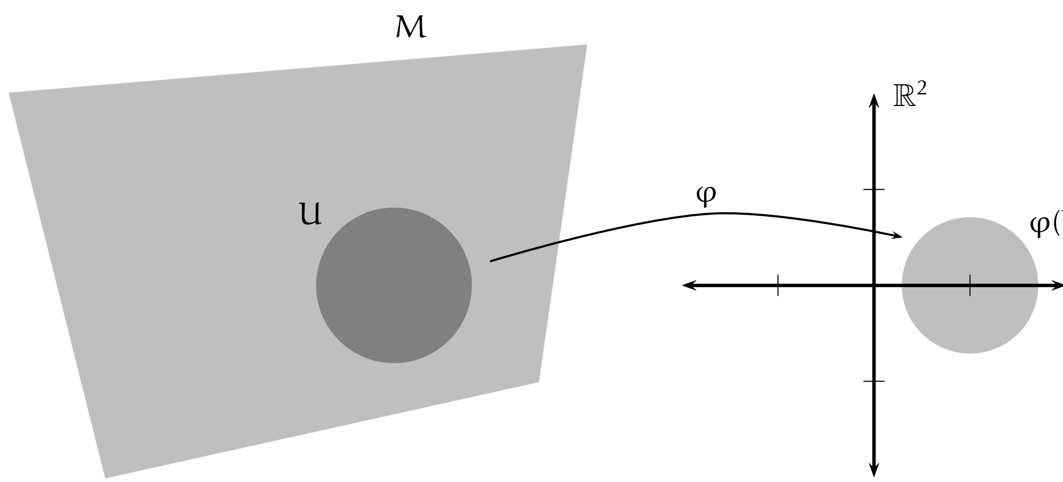

Figure 15.11.1: This illustrates a chart (U, φ) for the manifold M. It is a homeomorphism from the open set $U \subset M$ onto the open subset $\varphi(U) \subset \mathbb{R}^2$.

15.12 Examples of Charts

In a sense, a manifold is a generalization of the graph of a function from $\mathbb{R}^m$ to $\mathbb{R}$, just as a curve generalizes a function from $\mathbb{R}$ to $\mathbb{R}$.

► Example 15.12.1: Chart for a Function. Let $f : \mathbb{R}^m \to \mathbb{R}$. The manifold M is the graph of f,

$$M = \{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \mathbb{R}^m\}.$$

Define $F(\mathbf{x}) = \big(\mathbf{x}, f(\mathbf{x})\big)$. Here F maps $\mathbb{R}^m$ onto the graph of f. We only need one chart for this manifold. Let $\pi(\mathbf{x}, y) = \mathbf{x}$, which projects the graph to its $\mathbf{x}$ coordinates. The chart is (M, π) and $\pi^{-1} = F$. ◄

► Example 15.12.2: Charts for a Circle. The function φ_1 in Example 15.10.1 is a chart. Let $U_2 = \{(x, y) : y < 0\}$ and define φ_2 on $U_2 \cap C$ by $\varphi_2(x, y) = x$. This yields a chart with inverse g_2. Similarly, you can define open sets $U_3 = \{(x, y) : x > 0\}$ and $U_4 = \{(x, y) : x < 0\}$. The projections $\varphi_i : U_i \cap C$ defined for $i = 3, 4$ by $\varphi_i(x, y) = y$ are charts with inverses g_3 and g_4. ◄

This method can be made more general by using the Implicit Function Theorem.

► Example 15.12.3: Regular Curves have Charts. Corollary 15.5.2 shows how the same process can be used for any regular curve C defined as a level set of a $\mathcal{C}^1$ function $G(x, y)$. For any $(x_0, y_0) \in C$, it gives us an open set U. Depending on the case we are in, we define $\varphi(x, y) = x$ or $\varphi(x, y) = y$. The inverse on $\varphi(U)$ is $\varphi^{-1}(x) = \big(x, y(x)\big)$ or $\varphi^{-1}(x) = \big(x(y), y\big)$, respectively. ◄

15.13 Charts, Atlases, and Manifolds

Charts (coordinate systems) are bundled into an *atlas*, which can be used to give the manifold a differentiable structure.

Atlas. An atlas of a manifold M is an indexed family of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ that covers M in the sense that $\cup_{\alpha\in A} U_\alpha = M$. Let $W_{\alpha\beta} = U_\alpha \cap U_\beta$. An atlas is a $\mathcal{C}^k$ *atlas* if each *transition map* $\varphi_\beta \circ \varphi_\alpha^{-1}$ is $\mathcal{C}^k$ as a map from $\varphi_\alpha(W_{\alpha\beta})$ to $\varphi_\beta(W_{\alpha\beta})$, whenever $W_{\alpha\beta}$ is non-empty.

Notice that $\varphi_\alpha(W_{\alpha\beta}) \subset \varphi_\alpha(U_\alpha) \subset \mathbb{R}^m$ and $\varphi_\beta(W_{\alpha\beta}) \subset \varphi_\beta(U_\beta) \subset \mathbb{R}^m$, so it makes sense to consider whether transition functions are differentiable.

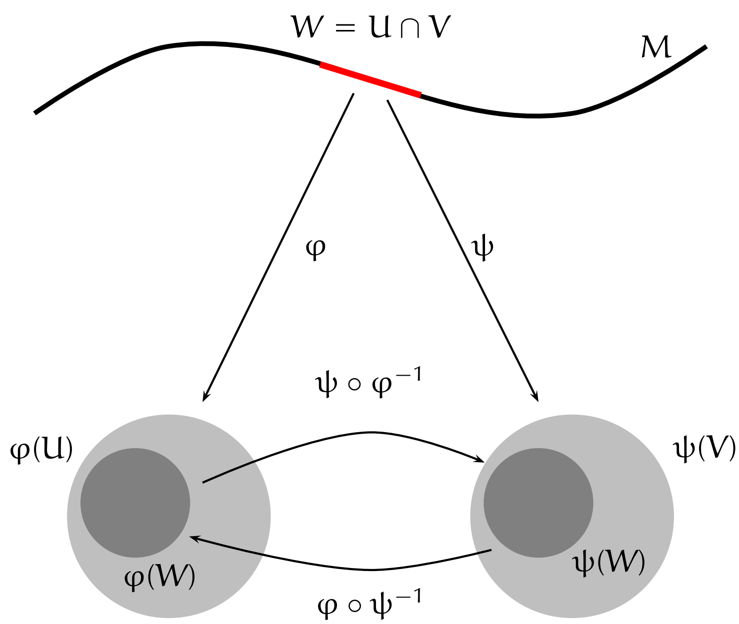

Figure 15.13.1: Here (φ, U) and (ψ, V) are charts with non-empty common domain $W = U \cap V$. The transition maps $\varphi\circ\psi^{-1}$ and $\psi\circ\varphi^{-1}$ are indicated. Here $\varphi\circ\psi^{-1}: \psi(W) \to \varphi(W)$ and $\psi \circ \varphi^{-1}: \varphi(W) \to \psi(W)$. The dark regions indicate the sets $\varphi(W) \subset \varphi(U)$ and $\psi(W) \subset \psi(V)$.

15.14 Differentiable Manifolds

We are now ready to define a differentiable manifold.

Differentiable Manifold. A $\mathcal{C}^1$ or *differentiable manifold* is a manifold where all of the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are $\mathcal{C}^1$. More generally, a manifold is a $\mathcal{C}^k$ *manifold* if all of the transition maps are $\mathcal{C}^k$ functions. When $k = 0$, we have a *continuous manifold*, where all of the transition maps are continuous ($\mathcal{C}^0$).

We can use the charts to define differentiable functions from one manifold to another.

Let M be a differentiable k-manifold and N be a differentiable m-manifold. We say a function $f\colon M \to N$ is *differentiable* at $\mathbf{x} \in M$ if there exist coordinate charts (U, φ) with $\mathbf{x} \in U$ for M and (V, ψ) with $f(\mathbf{x}) \in V$ for N such that

$$\psi \circ (f \circ \varphi^{-1})$$

is differentiable where it makes sense, meaning on

$$\varphi\big(f^{-1}(V) \cap U\big).$$

By using transition functions and the Chain Rule, it is easy to see that all charts containing $\mathbf{x}$ and $f(\mathbf{x})$ agree on the differentiability of f.

► **Example 15.14.1: Transition Maps on the Circle.** We continue to apply these ideas to the circle $C = \{(x, y) : x^2 + y^2 = 25\}$. The charts and their inverse mappings were introduced in Examples 15.10.1 and 15.12.2. Consider the charts (U_1, φ_1) and (U_3, φ_3). The intersection $\varphi_1^{-1}(U_1) \cap \varphi_3^{-1}(U_2)$ is the NE quadrant of the circle, $\{(x, y) \in C : x > 0, y > 0\}$. The transition functions for this pair of charts are $\varphi_3 \circ \varphi_1^{-1} = \sqrt{25 - x^2}$, defined for $x \in (0, 5)$ and $\varphi_1 \circ \varphi_3^{-1} = \sqrt{25 - y^2}$, defined for $y \in (0, 5)$. Both are $\mathcal{C}^\infty$, as are all of the other transition functions. This means C is a $\mathcal{C}^\infty$ manifold. ◄

15.15 Dimension of a Manifold

We start by showing that charts containing the same point must map to the same $\mathbb{R}^m$. This allows us to unambiguously define the dimension of the manifold M at each point $x \in M$. If the charts containing x all map to $\mathbb{R}^m$, we say that M *has dimension* m *at* x.

By using Lemma 15.9.1, we can show that every chart containing a point x must map to the same $\mathbb{R}^m$.

Theorem 15.15.1. *If $x \in U$ for some chart (U, φ) with $\varphi : U \to \mathbb{R}^k$ and $x \in V$ for some chart (V, ψ) with $\psi \colon V \to \mathbb{R}^m$, then $k = m$.*

Proof. Now $x \in U \cap V$, which is open. Consider the mapping $\varphi \circ \psi^{-1}$ which is defined on the open set $\psi(U \cap V)$.

$$\psi(U \cap V) \xrightarrow{\psi^{-1}} U \cap V \xrightarrow{\varphi} \varphi(U \cap V) \subset \mathbb{R}^k.$$

This is a homeomorphism between $\psi(U \cap V) \subset \mathbb{R}^m$ and the open set $\varphi(U \cap V) \subset \mathbb{R}^k$.

Lemma 15.9.1 now shows that $k = m$. ∎

If the dimension of M is m at all points of M, then the manifold M is m-dimensional. It's easy to show that connected manifolds must have the same dimension at every point.

15.16 Manifold or Not?

Graphs provide some of the simplest manifolds.

▶ **Example 15.16.1: An Atlas of One Chart.** Sometimes an atlas only needs one chart. Recall Example 15.12.1, where f is a real-valued function on $\mathbb{R}^m$ and M is its graph. We defined a single chart, (M, π) where $\pi(\mathbf{x}, y) = \mathbf{x}$ with inverse $F(\mathbf{x}) = \big(\mathbf{x}, f(\mathbf{x})\big)$. Since the single chart covers M, it suffices to define an atlas for M. ◀

Circles require more than one chart. One way to see this is that any circle is compact, so its image is also compact, and cannot be an open set in $\mathbb{R}^m$. The charts we've used give us an atlas for 4 charts.

▶ **Example 15.16.2: An Atlas for every Circle.** Take the circle C about the origin with radius 5, $C \cap U_1 \cap U_3$ is the upper right quadrant of the circle and $\varphi_3^{-1} U_1 = \{y : 5 > y > 0\}$. Then

$$\varphi_1 \circ \varphi_3^{-1}(y) = \varphi_1(x_0 + \sqrt{25 - y^2}, y) = x_0 + \sqrt{25 - y^2}.$$

This is $\mathcal{C}^1$ on $\{y : 5 > y > 0\}$. Similarly, the other transition maps are $\mathcal{C}^1$. This atlas makes C a $\mathcal{C}^1$ manifold. In fact, it makes C a $\mathcal{C}^\infty$ manifold.

This same technique can be used to define an atlas for any circle. In fact the technique can also be used on spheres, although it requires more charts, $2(m + 1)$ for the unit m-sphere, which is $S^m = \{\mathbf{x} \in \mathbb{R}^{m+1} : \|\mathbf{x}\|_2 = 1\}$. ◀

In contrast, the curve with a cusp in Example 14.16.1 is not a differentiable manifold as it is impossible to define a chart in a neighborhood of the origin that is compatible with a $\mathcal{C}^1$ atlas. This is due to the fact that the curve is not regular at the origin.

Curves that cross or have a T-intersection fail to even be continuous manifolds. Let x be the cross or tee point and (U, φ) a chart containing x. Choose U small enough that the ends of the cross or tee are not connected. If we remove x, the manifold is divided into 3 (tee) or 4 (cross) components, but $\varphi(U) \subset \mathbb{R}$ is a in interval, and removing x leaves two components. They cannot be homeomorphic sets.

If a manifold M is connected, the fact that the functions φ_α are homeomorphisms ensures that all of the charts in an atlas have the same dimension, which we define as the dimension of the manifold M. Manifolds that are not connected can have pieces of different dimensions.

▶ **Example 15.16.3: Regular Curves are Manifolds.** We saw how to construct charts for any regular curve in Example 15.12.3. Since this construction can be done at any point of C, the charts defined this way form an atlas for C. This shows that our original definition of a manifold is encompassed in the second definition using charts and an atlas. ◀

A similar procedure works for m-dimensional manifolds, but we need to beef up the Implicit Function Theorem to show how..

15.17 The Multidimensional Implicit Function Theorem

The Implicit Function Theorem can be extended to systems of k implicit functions of m variables.

Implicit Function Theorem. *Let $\mathbf{g}: \mathbb{R}^{k+m} \to \mathbb{R}^k$ be $\mathcal{C}^1$. For $\mathbf{y} \in \mathbb{R}^k$ and $\mathbf{x} \in \mathbb{R}^m$, we write $\mathbf{g}(\mathbf{y}, \mathbf{x})$. Consider the system of k equations*

$$\mathbf{g}(\mathbf{y}, \mathbf{x}) = \mathbf{c}.$$

If $\mathbf{g}(\mathbf{y}^*, \mathbf{x}^*) = \mathbf{c}$ and the $k \times k$ matrix $(D_{\mathbf{y}}\mathbf{g})(\mathbf{y}^*, \mathbf{x}^*)$ is invertible, then there is a $\mathcal{C}^1$ function $\hat{\mathbf{y}}: \mathbb{R}^m \to \mathbb{R}^k$ defined on some ball $B_r(\mathbf{x}^*) \subset \mathbb{R}^m$ with $r > 0$ such that

$$\mathbf{g}\big(\hat{\mathbf{y}}(\mathbf{x}), \mathbf{x}\big) = \mathbf{c} \tag{15.17.2}$$

for all $\mathbf{x} \in B_r(\mathbf{x}^*)$ and

$$\mathbf{y}^* = \hat{\mathbf{y}}(\mathbf{x}^*).$$

Moreover,

$$D\hat{\mathbf{y}}(\mathbf{x}^*) = -\big[(D_{\mathbf{y}}\mathbf{g})(\mathbf{y}^*, \mathbf{x}^*)\big]^{-1}(D_{\mathbf{x}}\mathbf{g})(\mathbf{y}^*, \mathbf{x}^*) \tag{15.17.3}$$

Proof. Omitted. ∎

We will not prove this version of the Implicit Function Theorem. The proof is rather more involved than the $\mathbb{R}^2$ version. As with the Implicit Function Theorem for $\mathbb{R}^2$, the hard part is finding the $\mathcal{C}^1$ function $\hat{\mathbf{y}}$. Once that is done, the Chain Rule can be applied to equation (15.17.2) obtain equation (15.17.3).

15.18 Manifolds as Solutions to Equation Systems

We start by defining regular points of systems of equations

Regular Point and Curves. A point $\mathbf{x}_0$ is a *regular point* of a $\mathcal{C}^1$ function $\mathbf{g}\colon \mathbb{R}^{k+m} \to \mathbb{R}^k$ if $\operatorname{rank} D\mathbf{g}(\mathbf{x}_0) = k$. If every point of $M = \{\mathbf{x} \in \mathbb{R}^{k+m} : \mathbf{g}(\mathbf{x}) = \mathbf{c}\}$ is a regular point, we say that M is a *regular* m*-manifold*.

We will show that every regular m-manifold has an atlas, justifying calling them manifolds.

Theorem 15.18.1. *Suppose* $\mathbf{g}\colon \mathbb{R}^{k+m} \to \mathbb{R}^k$ *is a* $\mathcal{C}^1$ *function and* $M = \{\mathbf{z} \in \mathbb{R}^{k+m} : \mathbf{g}(\mathbf{z}) = \mathbf{c}\}$ *is a regular* m*-manifold. Then for every point* $\mathbf{z}^*$ *there is an open set* U *and a function* $\varphi\colon M \to \mathbb{R}^m$ *such that* (U, φ) *is a chart. These charts form a* $\mathcal{C}^1$ *atlas.*

Proof. Since $\operatorname{rank} D\mathbf{g}(\mathbf{x}^*) = k$, we can divide the variables into two groups, $\mathbf{y} = (z_{i_1}, \dots, z_{i_k})$ and $\mathbf{x} = (z_{i_{k+1}}, \dots, z_{i_{k+m}})$ so that $D_{\mathbf{y}}\mathbf{g}(\mathbf{y}^*, \mathbf{x}^*)$ is invertible. The Implicit Function Theorem yields a $\mathcal{C}^1$ function $\hat{\mathbf{y}}\colon \mathbb{R}^m \to \mathbb{R}^k$ defined on ball $B_r(\mathbf{x}^*) \in \mathbb{R}^m$ with $r > 0$ such that $\mathbf{g}\big(\hat{\mathbf{y}}(\mathbf{x}), \mathbf{x}\big) = \mathbf{c}$. Set $U = B_r(\mathbf{y}^*, \mathbf{x}^*) \cap M$ and define $\varphi(\mathbf{y}, \mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in U$.

I claim (U, φ) is a chart. Both φ and $\varphi^{-1} = \big(\hat{\mathbf{y}}(\mathbf{x}), \mathbf{x}\big)$ are $\mathcal{C}^1$ functions and φ is a homeomorphism between U and the open set $V = \varphi^{-1}(U) \subset M$. Thus (U, φ) is a chart. Since M is regular, we can generate a chart containing any point of M, showing that the charts cover M.

All that is left is to show that the transition functions are $\mathcal{C}^1$. Now suppose we have two charts (U, φ) and (V, ψ) where $U \cap V$ is non-empty. Now consider

$$\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V).$$

Since both φ^{-1} and ψ are $\mathcal{C}^1$, the Chain Rule tells us that the transition maps are $\mathcal{C}^1$, showing that we have a $\mathcal{C}^1$ atlas. ∎

15.19 Tangent Spaces Revisited

As with curves, we can define tangents by considering tangents of curves in M. Let $\boldsymbol{z}(t)$ be a $\mathcal{C}^1$ curve with $\boldsymbol{z}(0) = \boldsymbol{z}_0$ and $\boldsymbol{z}'(0) = \boldsymbol{v}$, then $\mathbf{g}\big(\boldsymbol{z}(t)\big) = \mathbf{c}$. By the Chain Rule, $[D\mathbf{g}(\boldsymbol{z}_0)]\boldsymbol{v} = \mathbf{0}$, again showing any tangent at $\boldsymbol{x}_0$ is in the null space of $[D\mathbf{g}(\boldsymbol{z}_0)]$.

► Example 15.19.1: Isoquants. Suppose $f\colon \mathbb{R}^m \to \mathbb{R}$ is a production function. We normally assume $Df \gg \mathbf{0}$, so Df has rank one. By the Fundamental Theorem of Linear Algebra, $m = \operatorname{rank} Dg + \dim\ker Dg$, so $\dim\ker Dg = m-1$. A basis for the tangent space ($\ker Dg$) can be constructed by considering $\Delta x_1 \boldsymbol{e}_1 + \Delta x_i \boldsymbol{e}_i$. Then

$$\frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_i}\Delta x_i = 0.$$

It follows that

$$\frac{\Delta x_i}{\Delta x_1} = -\frac{f_1}{f_i} = -\,\mathrm{MRTS}_{1i}\,.$$

The slopes of the isoquant in various directions is given by the marginal rate of technical substitution. ◄

15.20 The Inverse Function Theorem

We close the chapter with the Inverse Function Theorem, which gives conditions that ensure a function will have a $\mathcal{C}^1$ inverse.

The Inverse Function Theorem. *Let* $\mathbf{f}\colon \mathbb{R}^m \to \mathbb{R}^m$ *be* $\mathcal{C}^1$. *If there is* $\mathbf{y}^*$ *with* $f(\mathbf{y}^*) = \mathbf{x}^*$ *and the* $m \times m$ *matrix* $D_{\mathbf{y}}\mathbf{f}(\mathbf{x}^*)$ *is invertible, then there is a* $\mathcal{C}^1$ *function* $\hat{\mathbf{y}}\colon \mathbb{R}^m \to \mathbb{R}^m$ *defined on some ball* $B_r(\mathbf{x}^*) \subset \mathbb{R}^m$ *with* $r > 0$ *and an open set* V *containing* $\mathbf{y}*$ *such that* $\mathbf{f}$ *is a bijection between* $B_r(\mathbf{x}^*)$ *and* V. *The inverse map* $\mathbf{f}^{-1} : V \to B_r(\mathbf{x}^*)$ *is also* $\mathcal{C}^1$ *with*

$$\mathbf{f}\big(\mathbf{f}^{-1}(\mathbf{x})\big) = \mathbf{x} \tag{15.20.4}$$

for all $\mathbf{x} \in B_r(\mathbf{x}^*)$. *Moreover,*

$$D\mathbf{f}^{-1}\big(\mathbf{f}(\mathbf{x}^*)\big) = -\big[D_{\mathbf{x}}\mathbf{f}(\mathbf{x}^*)\big]^{-1} \tag{15.20.5}$$

Proof. Set $\mathbf{g}(\mathbf{y},\mathbf{x}) = f(\mathbf{y}) - \mathbf{x}$, and $\mathbf{c} = 0$. Then $\mathbf{g}\colon \mathbb{R}^{2m} \to \mathbb{R}^m$, and $(D_{\mathbf{y}}\mathbf{g})^{-1} = (D\mathbf{f})^{-1}$ is invertible.

Then apply the Inverse Function Theorem to obtain a $\mathcal{C}^1$ function $\hat{\mathbf{y}}(\mathbf{x})$ that obeys $\mathbf{f}\big(\hat{\mathbf{y}}(\mathbf{x})\big)$ on $B_r(\mathbf{x})$, meaning that $\hat{\mathbf{y}} = \mathbf{f}^{-1}$ on $B_r(\mathbf{x})$. Apply the Chain Rule to equation (15.20.4) to obtain equation (15.20.5). ∎

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