

15. Implicit Functions and Their Derivatives

When y is written as a function of (x_1, \dots, x_m) ,

$$y = f(x_1, \dots, x_m)$$

we say that y is an *explicit function* of (x_1, \dots, x_m) .

Things are different when y and (x_1, \dots, x_m) are combined in a single function so that

$$f(x_1, \dots, x_m, y) = 0. \quad (15.0.1)$$

If the x_1, \dots, x_m determine y in equation (15.0.1), we say that y is an *implicit function* of (x_1, \dots, x_m)

With luck, we will be able to solve for y in terms of (x_1, \dots, x_m) . But that is not always possible. For example, the quintic equation

$$y^5 - 5xy + 4x^2 = 0$$

does not have an explicit solution, although we can say that $(x, y) = (1, 1)$ is a solution, as is $(1/4, 1)$, suggesting that $y(1) = 1$ and $y(1/4) = 1$. It's also clear that $y(0) = 0$. There are hints of a function here, but we can't solve for it.

When the equation implicitly defines y in terms of x , but we cannot write an expression for $y(x)$, we might still be able to determine the derivatives. The Implicit Function Theorem gives conditions for finding local functions for y and their derivatives.

15.1 Is there an Implicit Function?

One issue with equation (15.0.1) is that it is difficult to determine whether there even is an implicit function.

► **Example 15.1.1: No Implicit Function for a Circle.** Consider the equation $x^2 + y^2 = 25$. We know this has solutions such as $(0, 5)$ and $(3, 4)$. Does this expression implicitly define a function $y(x)$? In this case we can solve for y , obtaining

$$y(x) = \pm\sqrt{25 - x^2}.$$

There is a problem here. This is not a function!

Just look at the graph. For every x , $-5 < x < 5$, there are **two** values of y , not one. It's not a function.

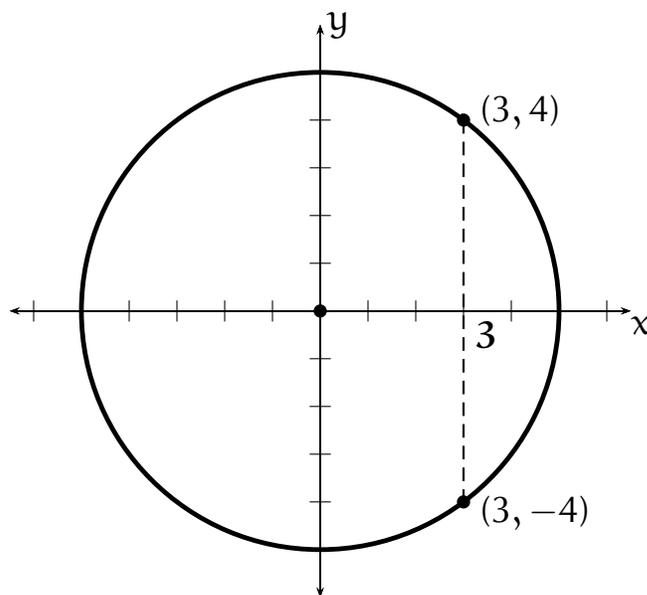


Figure 15.1.2: The circle is the graph of $x^2 + y^2 = 25$, which tries to implicitly define y as a function of x . As you can see, there are two solutions $y(x)$ for most values of x . This is illustrated at $x = 3$.

For every value of $x \in (-5, +5)$, there are two values of $y(x)$, not one. Only at $x = \pm 5$ do we have a function. This is illustrated in Figure 15.2.2. ◀

15.2 Picking an Implicit Function I

One way to work around this is to lower the bar, to give up the search for a global function and focus on a locally defined implicit function. We look for a function that solves the equation in a neighborhood of a point (x_0, y_0) . We can use one function near $(3, 4)$ and perhaps a different one near $(3, -4)$.

► **Example 15.2.1: Local Implicit Functions on a Circle I.** Here $y = +(25 - x^2)^{1/2}$ is implicitly defined by the equation $x^2 + y^2 = 25$ and includes the starting point $(x_0, y_0) = (3, 4)$. It can be defined on open sets as large as $(-5, 5)$, as in the upper blue arc in the figure below. Similarly, if $(x_0, y_0) = (3, -4)$, the function $y = -(25 - x^2)^{1/2}$ works for $x \in (-5, 5)$ and yields the lower light blue arc. The green line segment indicates the points $(x, 0)$ where both functions are defined, the points with $-5 < x < 5$.

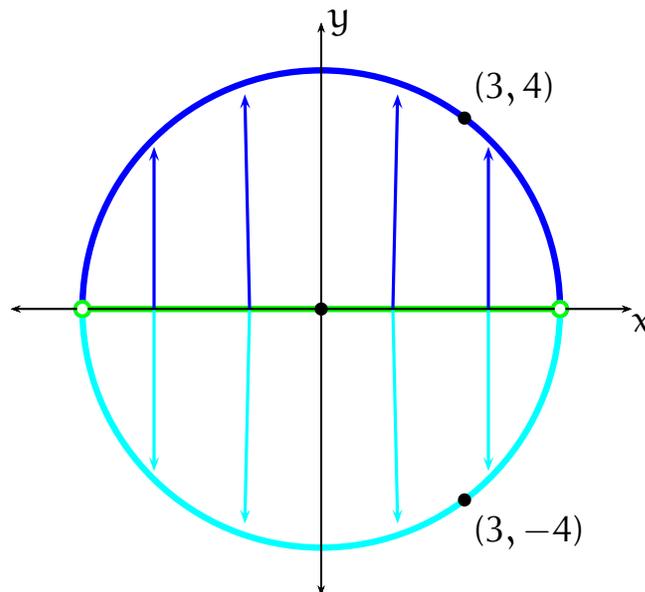


Figure 15.2.2: The circle is the graph of $x^2 + y^2 = 25$, which tries to implicitly define y as a function of x . We can define such a function on the upper half of the circle by $y = \sqrt{25 - x^2}$ for $-5 < x < 5$. This contains the point $(3, 4)$. The function $-\sqrt{25 - x^2}$ does the same thing for the lower half of the circle.



15.3 Picking an Implicit Function II

► **Example 15.3.1: Local Implicit Functions on a Circle II.** Another problem occurs at both $(5, 0)$ and $(-5, 0)$. Neither point allows us to define a function $y(x)$ on an open interval containing $x = \pm 5$. The points $x = \pm 5$ cannot be in the interior of the domain of y .

This is connected with the fact that the graph becomes vertical at those two points.

However, we can turn things around to make it work. Instead of defining y as function of x , we can define x as a function of y at those points. Indeed the functions $x_1(y) = \sqrt{25 - y^2}$ and $x_2(y) = -\sqrt{25 - y^2}$ do the trick. They can cover both of the red arcs, and more.

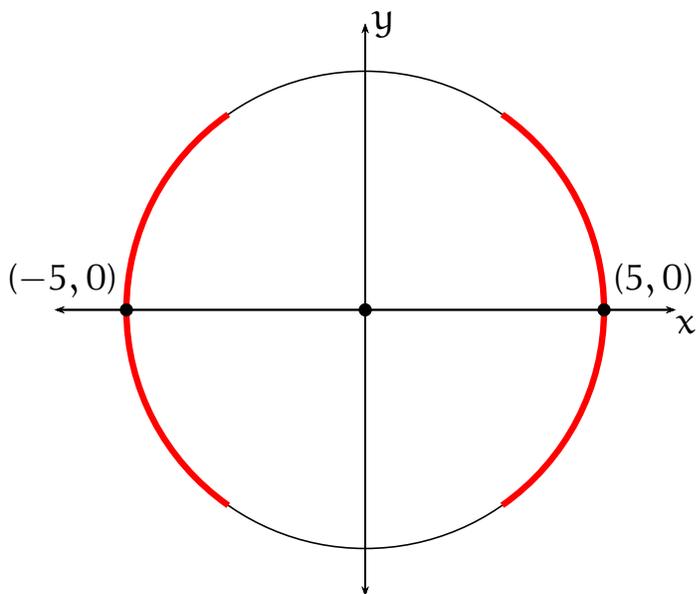


Figure 15.3.2: The circle is the graph of $x^2 + y^2 = 25$, which tries to implicitly define y as a function of x . The points $(5, 0)$ and $(-5, 0)$ pose particular problems as we are unable to write y as a function of x on a neighborhood of $x = \pm 5$ due to the verticality of the graph of y at $x = \pm 5$.



15.4 The Implicit Function Theorem for \mathbb{R}^2

The key result for implicit functions is the Implicit Function Theorem.¹ Here is a version for \mathbb{R}^2 . This is a special case of the general Implicit Function Theorem in section 15.30. Although the two-dimensional version is a bit easier to prove, it is difficult enough that we will not give it a separate proof. The proof of the multidimensional version of the Implicit Function Theorem (and related Inverse Function Theorem) will suffice.²

Implicit Function Theorem for \mathbb{R}^2 . Let $G(x, y)$ be a \mathcal{C}^1 function on a neighborhood of $(x_0, y_0) \in \mathbb{R}^2$. Suppose that $G(x_0, y_0) = c$. If

$$\frac{\partial G}{\partial y}(x_0, y_0) \neq 0,$$

there exists a \mathcal{C}^1 function $y(x)$ defined on an interval I containing x_0 such that:

- (a) $G(x, y(x)) = c$ for all $x \in I$,
- (b) $y(x_0) = y_0$, and
- (c) The function y obeys

$$y'(x_0) = -\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)}. \quad (15.4.2)$$

¹ The basic idea is already present in Newton in 1669. Leibniz's work includes an example of implicit differentiation in 1684. The theorem is generally attributed to Cauchy, who provided a rigorous statement and proof in two dimensions in his first Turin Memoir (1831).

² The history of the Implicit Function Theorem and more is covered in Steven G. Krantz and Harold R. Parks (2002), *The Implicit Function Theorem: History, Theory, and Applications*, Birkhäuser, Boston, Basel & Berlin.

15.5 More on the Implicit Function Theorem for \mathbb{R}^2

The condition $(\partial G/\partial y)(x_0, y_0) \neq 0$ rules out vertical graphs at (x_0, y_0) , making it possible to write y as a function of x . As long as that works, we're in business. We not only get y as a function of x , but if the original function was continuously differentiable, so is $y(x)$.

Point (c) follows from the Chain Rule once we establish that $y(x)$ is \mathcal{C}^1 . To see this, just differentiate $G(x, y(x)) = c$ to find

$$\frac{\partial G}{\partial x}(x_0, y_0) + \left[\frac{\partial G}{\partial y}(x_0, y_0) \right] y'(x_0) = 0.$$

Rearrange to obtain equation (15.4.2).

The statement of the theorem is similar to a 2-dimensional version of the Linear Implicit Function Theorem, except that it applies to differentiable functions, and only gives a local inverse, not a global inverse (the cost of generalizing to differentiable functions)

15.6 Using the Implicit Function Theorem

► **Example 15.6.1: The Implicit Function Theorem and the Circle.** How does this apply to the circle $x^2 + y^2 = 25$ we studied in Example 15.3.1? Here we set $G(x, y) = x^2 + y^2$ and $c = 25$. Let's try $(x_0, y_0) = (3, -4)$ and see what happens.

Here

$$\frac{\partial G}{\partial y}(3, -4) = -8 \neq 0,$$

so we can apply the Implicit Function Theorem to find $y(x)$ solving $x^2 + [y(x)]^2 = 25$ with $y(3) = -4$ and $y'(3) = -2(3)/2(-4) = 3/4$. Compare to the solution y_1 given by $y_1(x) = -(25 - x^2)^{1/2}$. Then

$$y_1'(x) = -(1/2)(25 - x^2)^{-1/2}(2x) = \frac{x}{\sqrt{25 - x^2}}.$$

Then $y_1'(3) = 3/4$, exactly as with y . ◀

The Implicit Function Theorem will be useful for writing one set of economic variables as a function of other variables. For instance, suppose we solve the consumer's problem for prices \mathbf{p} and income m . Can we write the demands for x and y as functions of prices and income? Once we characterize the solution via first order and second order equations, we will be able to use the Implicit Function Theorem to find whether we have proper demand functions.

15.7 One-dimensional Differentiable Manifolds

Regular Points and Curves. A point (x_0, y_0) is a *regular point* of a \mathcal{C}^1 function $G: \mathbb{R}^2 \rightarrow \mathbb{R}$ if either

$$\frac{\partial G}{\partial x}(x_0, y_0) \neq 0 \quad \text{or} \quad \frac{\partial G}{\partial y}(x_0, y_0) \neq 0.$$

If every point on $C = \{(x, y) : G(x, y) = c\}$ is a regular point, we say that C is a *regular curve*.

In the case of our circle, $G(x, y) = x^2 + y^2$ so $\partial G/\partial x = 2x$ and $\partial G/\partial y = 2y$. Since these can't both be zero on C , C is a regular curve.

Regular curves are examples of *one-dimensional differentiable manifolds*.³

³ The term “manifold” is a direct translation of Riemann’s term *Mannigfaltigkeit*, introduced in his Göttingen lecture of 1854, *Über die Hypothesen, welche der Geometrie zu Grundeliegen*, (*On the hypotheses that underlie geometry*). The lecture founded the field of differential geometry.

The German mathematician Bernhard Riemann (1826–1866) was one of the all-time greatest mathematicians. Only 39 when he died, he founded the field of differential geometry (later used by Einstein in his theory of General Relativity), proposed the Riemann Hypothesis, which is still being investigated and remains unproven after over 150 years. Some of his other accomplishments are developing the first rigorous definition of the integral, which he used to prove some results concerning Fourier series, and introducing Riemann surfaces in complex analysis.

15.8 Consequences of Regularity

In Theorem 15.9.1, we will show that regularity implies that at every point of (x, y) of C , we can either write y as a function of x or x as a function of y . We can handle horizontal segments by writing y as a function of x and vertical segments by writing x as a function of y .

This means that regular curves can twist and turn around. What they can't do is cross themselves. One curve that crosses itself is Bernoulli's Lemniscate, defined by the function⁴

$$G(x, y) = (x^2 + y^2) - 2a^2(x^2 - y^2) = 0.$$

We compute $DG(0, 0) = (0, 0)$, showing that the curve is not regular at the origin. In fact, there is no way to describe this curve that makes it regular at the origin.⁵

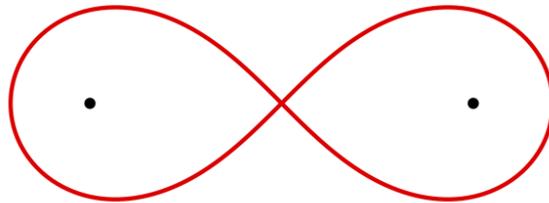


Figure 15.8.1: Bernoulli's Lemniscate: $(x^2 + y^2) - 2a^2(x^2 - y^2) = 0$.

⁴ This lemniscate is that of the Swiss mathematician Jacob (aka James or Jacques) Bernoulli (1655–1705 NS), one of the mathematicians in the Bernoulli family. He made a number of important contributions to mathematics. He discovered the constant e , base of the natural logarithms. He also provided the original formulation of the Law of Large Numbers. Finally, he and his brother Johann (1667–1748 NS) founded the Calculus of Variations, a method of optimization where the optimal point is a function rather than a number.

⁵ The only possibility is to be a one-dimensional manifold. If U is a neighborhood of the origin that is homeomorphic to an open interval, removing the origin breaks U into four components, but removing the corresponding point in the interval breaks it into two. The number of components of a set must be preserved under homeomorphism, so this is impossible.

15.9 Characterizing Regular Points

The following theorem characterizes the regular points of a curve. It follows immediately from the Implicit Function Theorem for \mathbb{R}^2 .

Theorem 15.9.1. *Let $G: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a \mathcal{C}^1 function. If (x_0, y_0) is a regular point on the curve $C = \{(x, y) : G(x, y) = c\}$, Then either*

1. $(\partial G / \partial y)(x_0, y_0) \neq 0$ and there is a \mathcal{C}^1 function $y(x)$ with $G(x, y(x)) = c$ on some neighborhood of (x_0, y_0) , or
2. $(\partial G / \partial x)(x_0, y_0) \neq 0$ and there is a \mathcal{C}^1 function $x(y)$ with $G(x(y), y) = c$ on some neighborhood of (x_0, y_0) .

Then either

$$y'(x_0) = -\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)} \quad \text{or} \quad x'(y_0) = -\frac{\frac{\partial G}{\partial y}(x_0, y_0)}{\frac{\partial G}{\partial x}(x_0, y_0)},$$

respectively.

When the curve defined by G is regular, we can strengthen this as follows.

Corollary 15.9.2. *Let $G: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a \mathcal{C}^1 function. If the curve $C = \{(x, y) : G(x, y) = c\}$ is regular, at every point (x_0, y_0) on the curve C , we can parameterize the curve by a \mathcal{C}^1 curve defined on an open set containing (x_0, y_0) : either as $(x, y(x))$ or $(x(y), y)$.*

The manifold C is considered one-dimensional since it can be locally described by a single parameter. It is differentiable because there are invertible differentiable functions that describe it locally, in a neighborhood of each point.

15.10 Tangent Spaces

What if we have a curve inside our manifold? Can we say anything about its tangent vector?

One way to define tangent vectors for a manifold M is to define them as the tangent vectors of all the curves in M . To that end, let

$$M = \{(x, y) \in \mathbb{R}^2 : G(x, y) = c\}$$

be a regular manifold and $\mathbf{x}(t) = (x(t), y(t))$ be a \mathcal{C}^1 curve in M . It obeys $G(x(t), y(t)) = c$ for all $t \in I$. Then we can consider $\mathbf{v} = \mathbf{x}'(t_0)$ to be a tangent vector to M at $\mathbf{x}(t_0) = (x(t_0), y(t_0))$.

Now $G(x(t), y(t)) = c$ for all t . We apply the Chain Rule at t_0 to find

$$[DG(x_0, y_0)](\mathbf{x}'(t_0), \mathbf{y}'(t_0)) = [DG(x_0, y_0)]\mathbf{v} = 0$$

since G is constant on the manifold. The tangent vector \mathbf{v} is in $\ker DG(x_0, y_0)$. In fact, any element of the kernel can be represented this way by appropriate choice of $(\mathbf{x}(t), \mathbf{y}(t))$.⁶ We call

$$T_{(x_0, y_0)}M = \ker DG(x_0, y_0)$$

the *tangent space* of M at (x_0, y_0) .

⁶ This will be easier to see once we introduce coordinate charts.

15.11 Tangent Space via the Gradient

Alternatively, we can relate the tangent space $T_{(x_0, y_0)}M$ to the gradient vector $\nabla G(x_0, y_0)$:

$$\nabla G(x_0, y_0) \cdot \mathbf{v} = 0.$$

The gradient gives the direction of fastest increase of G . The tangent space $T_{(x_0, y_0)}M$ is the set of all vectors perpendicular to it (definition 2).

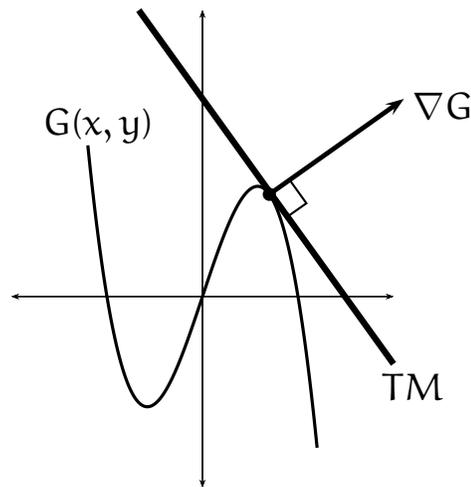


Figure 15.11.1: Here ∇G is the gradient and the tangent line to $G(x, y) = 0$, labelled TM , is shown by the heavy line perpendicular to the gradient vector. The function on the graph is $0 = G(x, y) = y + 3(x^3 - x)$.

15.12 m -Dimensional Manifolds

We've called regular curves 1-dimensional manifolds. But what is a manifold? We can give a general definition of a manifold by using homeomorphisms.⁷

Manifolds. A metric space M is a *manifold* if for every $x \in M$, there is a open neighborhood U of x and an integer $m \geq 0$ such that U is homeomorphic to an open subset of \mathbb{R}^m .

In other words, a manifold is a space that **locally** looks like m -dimensional Euclidean space, \mathbb{R}^m .

One advantage to this definition of a manifold is that it minimizes the baggage we carry about from \mathbb{R}^m . This allows us (forces us?) to define things in manifolds in a way that is independent of coordinates. If you check other sources, you will sometimes see manifolds defined as subsets of some ambient space \mathbb{R}^A . Manifolds do not have to sit inside some other space, although they often do.

Early on, manifolds were studied in ways that made it hard to distinguish intrinsic properties from those based on the coordinates currently in use. When Einstein proposed his theory of general relativity, he expressed everything in terms of coordinates, making it harder to see which concepts were dependent on the coordinate system used, and which were not, something that has been the bane of many students of general relativity.

The type of manifold we defined above is sometimes called a *topological manifold*. There are more specialized types of manifold, such as differentiable manifolds. Topological manifolds are the basic type of manifold. Other manifolds will be topological manifolds with additional properties, just as inner product spaces are vector spaces with additional properties. In the case of manifolds, we will mostly be interested in differentiable manifolds. These are topological manifolds which have differentiable structure added.

⁷ See page 1-1 of Michael Spivak, *A Comprehensive Introduction to Differential Geometry*, vol. 1, 1970. Also see page 1 of Morris Hirsch, *Differential Topology*, 1976. This definition can be extended to topological spaces, but that ideally involves paracompactness, which takes us too far afield.

15.13 Open Sets in \mathbb{R}^m are Manifolds

It's easy to find lots of manifolds. Any open set in \mathbb{R}^m is an m -dimensional manifold, including the entire space \mathbb{R}^m .

Theorem 15.13.1. *Let U be an open set in \mathbb{R}^m . Then U is an m -dimensional manifold.*

Proof. Let $x \in U$. Then U is a open neighborhood of x and U is trivially homeomorphic to itself (use the identity map id). As U is an open subset of \mathbb{R}^m , this means that every point $x \in U$ has a open neighborhood (U) that is homeomorphic to an open subset of \mathbb{R}^m (the same U). This shows that U is an m -dimensional manifold. ■

15.14 Circles are Manifolds I

Any circle in \mathbb{R}^2 is a one-dimensional manifold. We will show this for the circle defined by $x^2 + y^2 = 25$, but the methods used here apply to any circle. They can be easily generalized to spheres in \mathbb{R}^m .

We focus on the circle from Example 15.3.1, defined by $C = \{(x, y) : x^2 + y^2 = 25\}$. We will use four functions $g_i: V_i \rightarrow C$ with $V_i \subset \mathbb{R}$ that describe C as a manifold.

$$\begin{aligned} g_1(x) &= (x, \sqrt{25 - x^2}) && \text{for } x \in V_1 = (-5, 5) \\ g_2(x) &= (x, -\sqrt{25 - x^2}) && \text{for } x \in V_2 = (-5, 5) \\ g_3(y) &= (\sqrt{25 - y^2}, y) && \text{for } y \in V_3 = (-5, 5) \\ g_4(y) &= (-\sqrt{25 - y^2}, y) && \text{for } y \in V_4 = (-5, 5) \end{aligned}$$

The functions g_1 and g_2 describe the top and bottom halves of the circle, respectively. The functions for the right and left sides of the circle are g_3 and g_4 , respectively. I've labeled the domains V_i , which happen to be identical here, but need not be.

Every point except $(0, 5)$, $(0, -5)$, $(5, 0)$, and $(-5, 0)$ is in the range of exactly two of the g_i . Those points are in the range of only one of the g_i , with the points listed in the same order as g_1, \dots, g_4 .

The inverses of the g_i are the projections $\varphi_1(x, y) = \varphi_2(x, y) = x$ and $\varphi_3(x, y) = \varphi_4(x, y) = y$. That means that when $(x, y) \in C$, φ_1 can be written

$$\varphi_1(x, \sqrt{25 - x^2}) = x,$$

with similar definitions for the other $\varphi_u = i$. This guarantees that each of the four open half-circles is homeomorphic to the interval $(-5, 5)$. This shows that the circle C is a manifold.

The mappings g_1 and φ_1 are illustrated in Figure 15.15.1.

15.15 Circles are Manifolds II

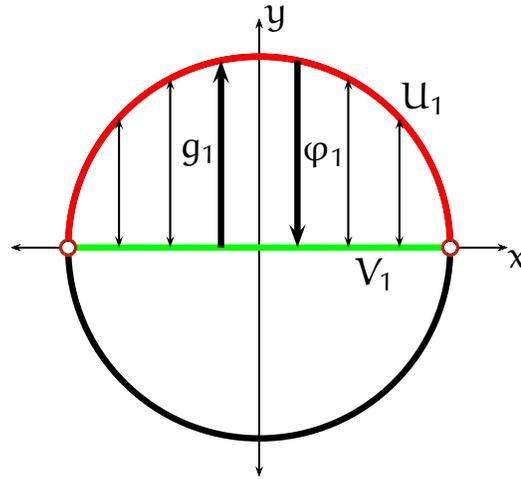


Figure 15.15.1: The vertical lines illustrate the bijection between U_1 (in red) and $V_1 = \{(x, 0) : |x| < 5\}$ (green) created by projection φ_1 onto the x -axis and g_1 , mapping back to the circle. Two examples are highlighted. One showing φ_1 mapping down to the x -axis, the other showing g_1 mapping up to the circle.

Here V_1 is embedded in \mathbb{R}^2 using the map $\psi(x) = (x, 0)$, with image $\{(x, 0) : x \in (-5, 5)\}$. The subspace topology ensures ψ is a homeomorphism.

The range of g_1 is the set $U_1 = \{(x, y) \in C : y > 0\}$. Then $g_1 \circ \varphi_1 : U_1 \rightarrow U_1$ and is defined by

$$g_1\left(\varphi_1(x, \sqrt{25 - x^2})\right) = g_1(x) = \left(x, \sqrt{25 - x^2}\right).$$

Here both φ_1 and its inverse g_1 are continuous, one-to-one, and onto. Therefore g_1 and φ_1 are homeomorphisms between U_1 and $V_1 = (-5, +5)$. Define U_2, U_3 , and U_4 similarly for the other g_i .⁸

⁸ Although the functions g_i can be defined on the closures of the intervals V_i , we would no longer have an open set as the range, and the functions cannot be \mathcal{C}^1 there.

15.16 Local Coordinate Systems

One difference between \mathbb{R}^m and manifolds is that we have a natural system of coordinates on \mathbb{R}^m , but none on any manifold. The mappings g_i allow us to establish local coordinate systems on our manifold C . The coordinates are dependent on the maps g_i . Different maps mean different coordinates.

Each $g_i: V_i \rightarrow U_i$ is a homeomorphism with inverse $\varphi_i: U_i \rightarrow V_i = (-5, 5)$. We will think of each φ_i as setting up a one-dimensional coordinate system in V_i for each of the U_i . There is only one coordinate in the system because it is a one-dimensional manifold.

The pair (U_i, φ_i) is called a *coordinate chart*, a term that is meant to remind you of nautical charts marked with latitude and longitude lines. The collection of all the charts on a manifold is called an *atlas*. To be an atlas M , every point of M must be contained in some chart. Since each U_i is homeomorphic to an open set $V_i \subset \mathbb{R}^m$, and M is covered by the charts, it is a manifold.⁹

⁹ The use of local coordinates such as these dates back at least to Gauss in 1827.

15.17 Coordinate Charts

As you have seen, we can set up coordinate system by using a system of coordinate charts. This allow us to describe open sets in a manifold M by means of a coordinate system in some \mathbb{R}^m .

Chart. A *chart* or *coordinate system* on a manifold M is a pair (U, φ) where U is a open subset of M and φ is a continuous homeomorphism from U to an open subset of some Euclidean space \mathbb{R}^m .

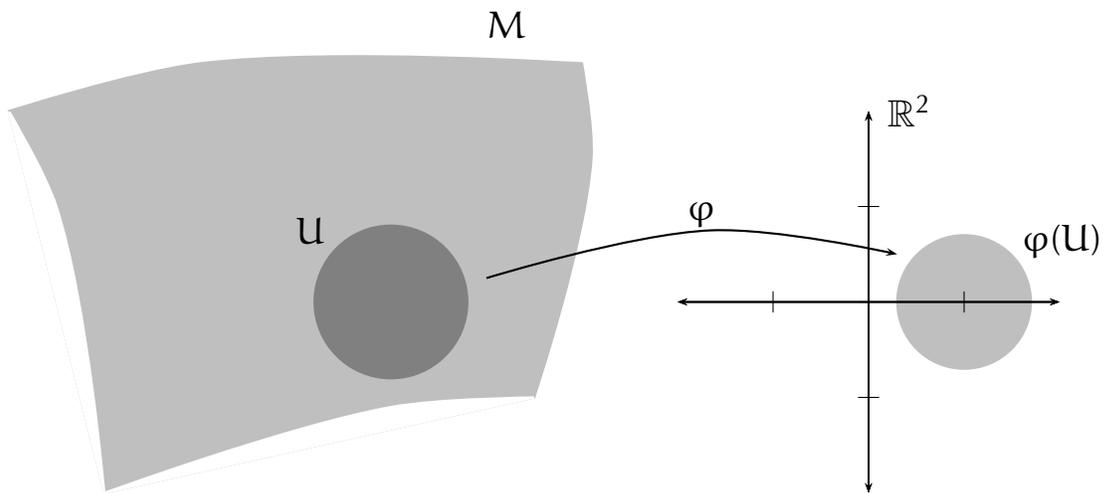


Figure 15.17.1: This illustrates a chart (U, φ) for the manifold M . It is a homeomorphism from the open set $U \subset M$ onto the open subset $\varphi(U) \subset \mathbb{R}^2$.

15.18 Reconciling Local Coordinate Systems

We now have two different coordinate systems that describe all but four points of the circle. How do they relate?

When both exist for a point, we can change the i coordinates to the j coordinates. To make this concrete, consider the point $(3, 4)$. This is in the domain of both φ_1 (top half) and φ_3 (right half). If we have the φ_1 coordinate, which is 3, we map it to $(3, 4)$, then project on the y -axis to obtain the φ_3 coordinate 4.

More generally, when $U_i \cap U_j$ is non-empty, the coordinate change takes coordinate $x \in \varphi_i(U_i \cap U_j) \in V_i$, maps it to $\varphi_i^{-1}(x) = g_i(x) \in U_i \cap U_j$, and then applies φ_j to map it to $\varphi_j(\varphi_i^{-1}(x)) \in \varphi_j(U_i \cap U_j) \in V_j$. In our example above, $U_i \cap U_j = C \cap \mathbb{R}_{++}^2$ and both $\varphi_i(U_i \cap U_j) \subset V_i$ and $\varphi_j(U_i \cap U_j) \subset V_j$ are the interval $(0, 5)$.

$$V_i \xrightarrow{\varphi_i^{-1}} U_i \cap U_j \xrightarrow{\varphi_j} V_j.$$

15.19 Transition Maps

In short, we change coordinates from φ_i to φ_j by applying the *transition map*

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(\mathcal{U}_i \cap \mathcal{U}_j) \subset V_i \rightarrow \varphi_j(\mathcal{U}_i \cap \mathcal{U}_j) \subset V_j. \quad (15.19.3)$$

Transition maps are only defined when $\mathcal{U}_i \cap \mathcal{U}_j$ is non-empty.

So what does this coordinate change map look like? On $\mathcal{U}_1 \cap \mathcal{U}_3$, the φ_1 coordinate $x \in (0, 5)$ is mapped to $\varphi_1^{-1}(x) = (x, \sqrt{25 - x^2})$. Then φ_3 maps that to $\sqrt{25 - x^2}$. This map is not only a homeomorphism on $\mathcal{U}_1 \cap \mathcal{U}_3 = (0, 5)$, but is actually \mathcal{C}^∞ there.

We will shortly define differentiable manifolds, and the defining feature will be that the transition maps are differentiable. We will need some more definitions first, but when we have them, we will find that the circle C , together with the differentiable structure given by the charts $\{g_i\}_{i=1}^4$ is a \mathcal{C}^∞ manifold!

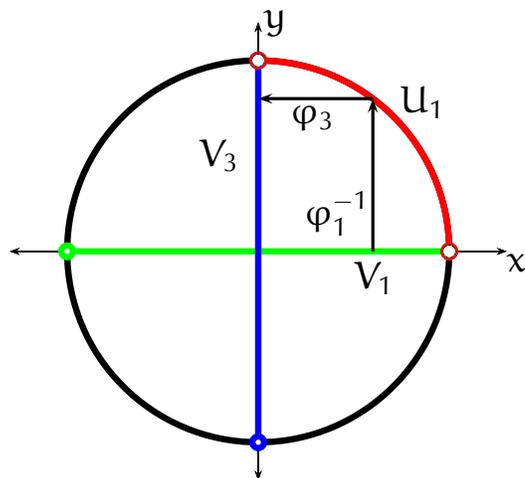


Figure 15.19.1: Here V_1 is marked in green and V_3 in blue and $\mathcal{U}_1 = \varphi_1^{-1}(V_1) \cap \varphi_3^{-1}(V_3)$ is shown in red. We follow the transition map $\varphi_3 \circ \varphi_1^{-1}$ from V_1 , up to \mathcal{U}_1 , and over to V_3 . This transition map is only defined on $\varphi_1^{-1}(\mathcal{U}_1)$, which is the right-hand portion of V_1 , not including $(0, 0)$.

15.20 Atlases of Charts

As mentioned before, we bundle our charts (coordinate systems) into an *atlas*, which can be used to give the manifold a differentiable structure via the transition maps.

Atlas. An *atlas* of a manifold M is an indexed family of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ that covers M ($M \subset \bigcup_{\alpha \in A} U_\alpha$).

Since the atlas covers M , every point in M is in the domain of at least one chart.

15.21 Smooth Atlases and Transition Maps

We now consider the smoothness of an atlas.

Transition Maps and \mathcal{C}^k Atlas. Given an atlas $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ let $W_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. An atlas is a \mathcal{C}^k atlas if each transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is \mathcal{C}^k as a map from $\varphi_\alpha(W_{\alpha\beta})$ to $\varphi_\beta(W_{\alpha\beta})$, whenever $W_{\alpha\beta}$ is non-empty.

Since $\varphi_\alpha(W_{\alpha\beta}) \subset \varphi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^m$ and $\varphi_\beta(W_{\alpha\beta}) \subset \varphi_\beta(\mathcal{U}_\beta) \subset \mathbb{R}^m$, every transition function maps a subset of \mathbb{R}^m into \mathbb{R}^m . That means it makes sense to consider whether transition functions are \mathcal{C}^k .

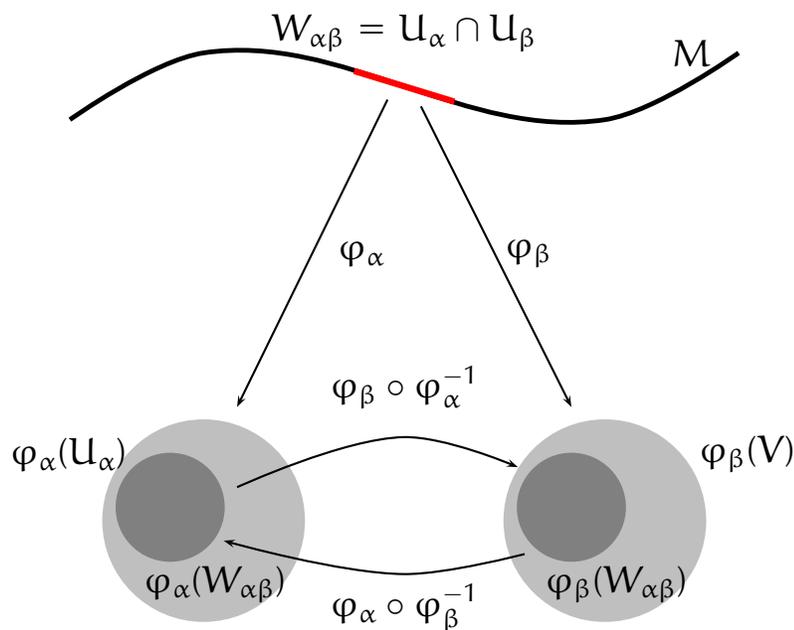


Figure 15.21.1: Here $(\varphi_\alpha, \mathcal{U})$ and $(\varphi_\beta, \mathcal{V})$ are charts with non-empty common domain $W_\beta = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. The transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ and $\varphi_\beta \circ \varphi_\alpha^{-1}$ are indicated. Here $\varphi_\alpha \circ \varphi_\beta^{-1}: \varphi_\beta(W) \rightarrow \varphi_\alpha(W)$ and $\varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(W) \rightarrow \varphi_\beta(W)$. The darker regions indicate the sets $\varphi_\alpha(W) \subset \varphi_\alpha(\mathcal{U})$ and $\varphi_\beta(W) \subset \varphi_\beta(\mathcal{V})$.

15.22 Differentiable Manifolds

We are finally ready to define a differentiable manifold.¹⁰

Differentiable Manifold. A \mathcal{C}^1 or *differentiable manifold* is a manifold where all of the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are \mathcal{C}^1 . More generally, a manifold is a \mathcal{C}^k *manifold* if all of the transition maps are \mathcal{C}^k functions. When $k = 0$, we have a *topological manifold*, where all of the transition maps are continuous, but need not be differentiable.

We continue to apply these ideas to the circle $C = \{(x, y) : x^2 + y^2 = 25\}$.

► **Example 15.22.1: Transition Maps on the Circle.** The charts and their inverse mappings were introduced in Section 15.14 and Example 15.26.1. Consider the charts (U_1, φ_1) and (U_3, φ_3) . The intersection $\varphi_1^{-1}(U_1) \cap \varphi_3^{-1}(U_2)$ is the NE quadrant of the circle, $\{(x, y) \in C : x > 0, y > 0\}$. The transition functions for this pair of charts are $\varphi_3 \circ \varphi_1^{-1} = \sqrt{25 - x^2}$, defined for $x \in (0, 5)$ and $\varphi_1 \circ \varphi_3^{-1} = \sqrt{25 - y^2}$, defined for $y \in (0, 5)$. Both are \mathcal{C}^∞ , as are all of the other transition functions. This means C is a not just a topological manifold, but a \mathcal{C}^∞ manifold. ◀

¹⁰ This definition, using transition maps, dates to O. Veblen and J.H.C. Whitehead (1931) "A set of axioms for differential geometry", *Proc. Nat. Acad. Sci.* **17**, 551–561.

Oswald Veblen (1880–1960) was an American mathematician who specialized in topology, differential geometry, and projective geometry. His uncle was the sociologist Thorstein Veblen (*Theory of the Leisure Class*).

J.H.C. Whitehead (1904–1960) was a British mathematician. During World War II, he applied operations research to submarine warfare, and later joined the codebreakers at Bletchley Park. In algebraic topology, he defined CW complexes and developed simple homotopy theory. The British mathematician and philosopher Alfred North Whitehead was his uncle, who is best known in mathematics for the three-volume *Principia Mathematica*, written with Bertrand Russell.

15.23 Dimension of a Manifold

We will show that charts containing the same point must map to the same \mathbb{R}^m . This allows us to unambiguously define the dimension of the manifold M at each point $x \in M$. If the charts containing x all map to \mathbb{R}^m , we say that M has dimension m at x .

By using Proposition 34.9.1, we can show that every chart containing a point x must map to the same \mathbb{R}^m .

Theorem 15.23.1. *If $x \in U$ for some chart (U, φ) with $\varphi : U \rightarrow \mathbb{R}^k$ and $x \in V$ for some chart (V, ψ) with $\psi : V \rightarrow \mathbb{R}^m$, then $k = m$.*

Proof. Now $x \in U \cap V$, which is open. Consider the mapping $\varphi \circ \psi^{-1}$ which is defined on the open set $\psi(U \cap V)$.

$$\psi(U \cap V) \xrightarrow{\psi^{-1}} U \cap V \xrightarrow{\varphi} \varphi(U \cap V) \subset \mathbb{R}^k.$$

This is a homeomorphism between $\psi(U \cap V) \subset \mathbb{R}^m$ and the open set $\varphi(U \cap V) \subset \mathbb{R}^k$.

Proposition 34.9.1 now shows that $k = m$. ■

If the dimension of M is m at all points of M , then the manifold M is m -dimensional. It's easy to show that connected manifolds must have the same dimension at every point.

15.24 Charts and Coordinates in Vector Spaces

A simple example of charts occurs in finite-dimensional vector spaces, where we use them to set up coordinate systems.

► **Example 15.24.1: Charts for Vector Spaces.** Suppose we have an m -dimensional vector space V with the usual topology. Let \mathcal{B}_1 and \mathcal{B}_2 be bases for V and \mathbf{B}_1 and \mathbf{B}_2 the corresponding basis matrices. We can now define two charts on V . The chart φ_i gives us the coordinates in \mathbb{R}^m . From Section 31.23 we know that $\varphi_i(\mathbf{x}) = \mathbf{B}_i^{-1}\mathbf{x} = \mathbf{t}$, the coordinates in \mathbb{R}^m . Both of these mappings, φ_1 and φ_2 , are one-to-one onto mappings from V to \mathbb{R}^m . Moreover, as linear mappings they are continuous. As both φ_i are onto, Either one by itself would form an atlas. Together, they are also an atlas.

Let's examine the transition maps. The two charts are (V, φ_1) and (V, φ_2) . The intersection of their domains is all of V and $\varphi_j \circ \varphi_i^{-1}(\mathbf{t}) = \mathbf{B}_j^{-1}\mathbf{B}_i\mathbf{t}$ for any $\mathbf{t} \in \mathbb{R}^m$. The transition map is exactly the coordinate change formula we used in both Equations 31.23.1 and 15.19.3. ◀

15.25 Graphs are Manifolds**11/03/22**

In a sense, a manifold is a generalization of the graph of a function from \mathbb{R}^m to \mathbb{R} , just as a curve generalizes a function from \mathbb{R} to \mathbb{R} .

► **Example 15.25.1: Graph of a Function.** Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$. Let M be the graph of f ,

$$M = \left\{ (\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \mathbb{R}^m \right\}.$$

The set M is an m dimensional manifold. Define $F(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}))$. Here F maps \mathbb{R}^m onto the graph of f . We only need one chart for this manifold. Let $\pi(\mathbf{x}, y) = \mathbf{x}$, which projects the graph onto its first m coordinates, \mathbf{x} . The chart is (M, π) and $\pi^{-1} = F$. The atlas is also (M, π) . There are no transition maps to worry about. ◀

15.26 Charts and Atlases for Regular Curves

We used charts when treating circles as manifolds.

► **Example 15.26.1: Charts for a Circle.** The function φ_1 in Section 15.14 is a chart. Let $U_2 = \{(x, y) : y < 0\}$ and define φ_2 on $U_2 \cap C$ by $\varphi_2(x, y) = x$. This yields a chart with inverse g_2 . Similarly, you can define open sets $U_3 = \{(x, y) : x > 0\}$ and $U_4 = \{(x, y) : x < 0\}$. The projections $\varphi_i : U_i \cap C$ defined for $i = 3, 4$ by $\varphi_i(x, y) = y$ are charts with inverses g_3 and g_4 . Together, the four charts cover C , so they form an atlas. Notice that any two would leave one of the points of C uncovered. ◀

This method can be made more general by using the Implicit Function Theorem.

► **Example 15.26.2: Regular Curves have Charts.** Corollary 15.9.2 shows how the same process can be used for any regular curve C defined as a level set of a \mathcal{C}^1 function $G(x, y)$. For any $(x_0, y_0) \in C$, it gives us an open set U . Depending on the case we are in, we define $\varphi(x, y) = x$ or $\varphi(x, y) = y$. The inverse on $\varphi(U)$ is $\varphi^{-1}(x) = (x, y(x))$ or $\varphi^{-1}(x) = (x(y), y)$, respectively.

Since the Implicit Function Theorem can be applied at any point of the curve C , for each point, there is a chart that covers it. The charts form an atlas. The transition functions are all \mathcal{C}^1 , so we have a \mathcal{C}^1 manifold. ◀

15.27 Tangent Spaces Revisited

As with curves, we can define tangents by considering tangents of curves in M . Since M is m -dimensional, we potentially have a much richer collection of tangents to study.

Let $\mathbf{z}(t)$ be a \mathcal{C}^1 curve with $\mathbf{z}(0) = \mathbf{z}_0$ and $\mathbf{z}'(0) = \mathbf{v}$, then $\mathbf{g}(\mathbf{z}(t)) = \mathbf{c}$. By the Chain Rule, $[\mathbf{Dg}(\mathbf{z}_0)]\mathbf{v} = \mathbf{0}$, again showing any tangent at \mathbf{z}_0 is in the null space of $[\mathbf{Dg}(\mathbf{z}_0)]$.

► **Example 15.27.1: Isoquants.** Suppose $f: \mathbb{R}^m \rightarrow \mathbb{R}$ is a production function. We normally assume $\mathbf{Df} \gg \mathbf{0}$, so \mathbf{Df} has rank one. By the Fundamental Theorem of Linear Algebra, $m = \text{rank } \mathbf{Df} + \dim \ker \mathbf{Df}$, so $\dim \ker \mathbf{Df} = m - 1$. A basis for the tangent space ($\ker \mathbf{Df}$) can be constructed by considering $\Delta x_1 \mathbf{e}_1 + \Delta x_i \mathbf{e}_i$. Then

$$\frac{\partial f}{\partial x_1} \Delta x_1 + \frac{\partial f}{\partial x_i} \Delta x_i = 0.$$

It follows that

$$\frac{\Delta x_i}{\Delta x_1} = -\frac{f_1}{f_i} = -\text{MRTS}_{1i}.$$

The slopes of the isoquant in various directions is given by the marginal rate of technical substitution. ◀

15.28 Differentiable Functions on Manifolds

We can't define differentiable functions directly on a manifold, as our manifold may not have the required vector space structure to define derivatives. However, by using charts to write coordinates in \mathbb{R}^m , we can define differentiable functions from one differentiable manifold to another.

Let M be a differentiable m -manifold and N be a differentiable n -manifold. We say a function $f: M \rightarrow N$ is *differentiable* at $\mathbf{x} \in M$ if there exist coordinate charts (U, φ) with $\mathbf{x} \in U \subset M$ and (V, ψ) with $f(\mathbf{x}) \in V \subset N$ such that

$$\psi \circ (f \circ \varphi^{-1})$$

is differentiable where it makes sense, meaning on the set $\varphi(U \cap f^{-1}(V))$. Consider the mapping

$$\varphi(U) \xrightarrow{\varphi^{-1}} U \subset M \xrightarrow{f} V \subset N \xrightarrow{\psi} \psi(V).$$

which is defined on

$$\varphi(U \cap f^{-1}(V)).$$

By using transition functions and the Chain Rule, it is easy to see that all charts containing \mathbf{x} and $f(\mathbf{x})$ agree on the differentiability of f . We can similarly define \mathcal{C}^k functions for any k .

As a sanity check, suppose M and N are open subsets $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^m$. The only charts are the respective identity maps, so if $f: M \rightarrow N$, we need only check whether $\text{id} \circ (f \circ \text{id}^{-1}) = f$ is differentiable as a function from U to V . I.e., it must be differentiable in the ordinary sense.

15.29 Differentiable Functions on the Manifold C

We'll illustrate how this works with a simple example mapping the circle C into the real line.

► **Example 15.29.1:** A \mathcal{C}^∞ function from C to \mathbb{R} . The atlas for C is given by the charts $(U_i, \varphi_i)_{i=1}^4$ defined in Section 15.14. Recall that $\varphi_i^{-1} = g_i$.

Consider the function $f(x, y) = x^2 + y^4$ with $f: C \rightarrow \mathbb{R}$. For \mathbb{R} , there is a single chart (\mathbb{R}, id) where $\text{id}(x) = x$ is the identity map.

We now compute $\text{id} \circ f \circ \varphi_i^{-1} = \text{id} \circ f \circ g_i$ for $(x, y) \in C$. Since id is the identity, this reduces to $f \circ g_i$. Now for $i = 1, 2$, we have

$$\text{id} \circ f \circ \varphi_i^{-1}(x) = g_{i1}(x)^2 + g_{i2}(x)^4 = x^2 + (25 - x^2)^2,$$

while for $i = 3, 4$,

$$\text{id} \circ f \circ \varphi_i^{-1}(y) = g_{i1}(y)^2 + g_{i2}(y)^4 = 25 - y^2 + y^4.$$

It is easy to see that these functions are not only differentiable, but \mathcal{C}^∞ , showing that f is a \mathcal{C}^∞ function from C to \mathbb{R} . ◀

15.30 The Multidimensional Implicit Function Theorem

So far, we have only examined one-dimensional manifolds using the Implicit Function Theorem. The Implicit Function Theorem can be extended to systems of k implicit functions of m variables. Then we can use the Implicit Function Theorem to describe many m -dimensional manifolds.¹¹

Implicit Function Theorem. Let $\mathbf{g} : \mathbb{R}^{m+k} \rightarrow \mathbb{R}^k$ be \mathcal{C}^1 . For $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^k$ we write $\mathbf{g}(\mathbf{x}, \mathbf{y})$. Consider the system of k equations

$$\mathbf{g}(\mathbf{x}, \mathbf{y}) = \mathbf{c}.$$

If $\mathbf{g}(\mathbf{x}^*, \mathbf{y}^*) = \mathbf{c}$ and the $k \times k$ matrix $(D_{\mathbf{y}}\mathbf{g})(\mathbf{x}^*, \mathbf{y}^*)$ is invertible, then there is a \mathcal{C}^1 function $\hat{\mathbf{y}} : \mathbb{R}^m \rightarrow \mathbb{R}^k$ defined on some ball $B_r(\mathbf{x}^*) \subset \mathbb{R}^m$ with $r > 0$ such that

$$\mathbf{g}(\mathbf{x}, \hat{\mathbf{y}}(\mathbf{x})) = \mathbf{c} \tag{15.30.4}$$

for all $\mathbf{x} \in B_r(\mathbf{x}^*)$ and

$$\mathbf{y}^* = \hat{\mathbf{y}}(\mathbf{x}^*).$$

Moreover,

$$D\hat{\mathbf{y}}(\mathbf{x}^*) = -[(D_{\mathbf{y}}\mathbf{g})(\mathbf{x}^*, \mathbf{y}^*)]^{-1}(D_{\mathbf{x}}\mathbf{g})(\mathbf{x}^*, \mathbf{y}^*). \tag{15.30.5}$$

As an exercise, it's worth considering how the Linear Implicit Function Theorem (section 7.35) relates to the Multidimensional Implicit Function Theorem. Is the linear theorem a special case of this one?

¹¹ The Italian mathematician Ulisse Dini (1845–1918) generalized the Implicit Function Theorem to m dimensions. Dini worked primarily in real analysis. Among other things, he developed a criterion for the pointwise convergence of Fourier series.

15.31 The Inverse Function Theorem

The Inverse and Implicit Function Theorems are closely related. It's fairly easy to prove one from the other. We will prove a multidimensional Implicit Function Theorem by first proving a multidimensional Inverse Function Theorem.

The Inverse Function Theorem gives conditions that ensure a function will have a \mathcal{C}^1 inverse. Of the two theorems, the proof of the Inverse Function Theorem is easier to understand, so it is the one we prove. After that, the Implicit Function Theorem follows easily.¹²

The Inverse Function Theorem. Let $f: \mathbb{R}^m \rightarrow \mathbb{R}^m$ be \mathcal{C}^1 . If there is \mathbf{y}^* with $f(\mathbf{y}^*) = \mathbf{x}^*$ and the $m \times m$ matrix $D_{\mathbf{y}}f(\mathbf{y}^*)$ is invertible, then there is a \mathcal{C}^1 function $\hat{\mathbf{y}}: \mathbb{R}^m \rightarrow \mathbb{R}^m$ defined on some ball $B_r(\mathbf{y}^*)$ and an open neighborhood V of \mathbf{x}^* where f is a bijection between $B_r(\mathbf{y}^*)$ and V . The inverse map $\hat{\mathbf{y}}: V \rightarrow B_r(\mathbf{y}^*)$ is also \mathcal{C}^1 with

$$f(\hat{\mathbf{y}}(\mathbf{x})) = \mathbf{x} \quad (15.31.6)$$

for all $\mathbf{x} \in V$. Moreover,

$$D\hat{\mathbf{y}}(\mathbf{x}^*) = [D_{\mathbf{y}}f(\mathbf{y}^*)]^{-1} \quad (15.31.7)$$

¹² The earliest version of the Inverse Function Theorem seems to be that of Joseph Louis Lagrange in 1770. It was later proved by Picard and Goursat via iteration. For us, the iteration is replaced by the Contraction Mapping Theorem.

The French mathematician Édouard Jean-Baptiste Goursat (1858–1936) is probably best known today for the Cauchy-Goursat Theorem of complex analysis. He was also one of the 19th century mathematicians who explored geometry in more than three dimensions. In his own day he was best known for his *Cours d'analyse mathématique*.

15.32 Proof of the Inverse Function Theorem I

Proof (Inverse Function Theorem). Let $\mathbf{T} = D_{\mathbf{y}}f(\mathbf{y}^*)$. By hypothesis, \mathbf{T} is invertible. Taylor's formula yields $f(\mathbf{y}) = f(\mathbf{y}^*) + \mathbf{T}(\mathbf{y} - \mathbf{y}^*) + R_1$. We can approximate $\mathbf{x} = f(\mathbf{y})$ by dropping the remainder R_1 and using

$$\mathbf{x} = f(\mathbf{y}^*) + \mathbf{T}(\mathbf{y} - \mathbf{y}^*).$$

Multiplying by \mathbf{T}^{-1} and rearranging, we obtain

$$\mathbf{y} = \mathbf{y}^* + \mathbf{T}^{-1}(\mathbf{x} - f(\mathbf{y}^*)).$$

This equation approximates \mathbf{y} , which obeys $f(\mathbf{y}) = \mathbf{x}$. If we knew $f^{-1}(\mathbf{x})$ existed, we would be approximating it.

Define a new function $F_{\mathbf{x}}(\mathbf{y})$ by replacing \mathbf{y}^* with \mathbf{y} in the right hand side:

$$F_{\mathbf{x}}(\mathbf{y}) = \mathbf{y} + \mathbf{T}^{-1}(\mathbf{x} - f(\mathbf{y})).$$

The proof uses a form of Newton's method to estimate the inverse. We normally think of iterating Newton's method to increase the estimate's precision. Instead of iterating, we apply the Contraction Mapping Theorem. The iteration is then hidden in its proof. We'll show that the mapping from \mathbf{y} to $F_{\mathbf{x}}(\mathbf{y})$ is a contraction for each \mathbf{x} . As such, it has a fixed point which will provide the inverse function.

The proof also uses some facts about matrix norms that we establish in sections 35–25.

Proof continues ...

15.33 Proof of the Inverse Function Theorem II

Proof continues. By continuity of $D_{\mathbf{y}}\mathbf{f}$, we may choose an $r > 0$ small enough that for $\mathbf{y} \in B_r(\mathbf{y}^*)$,

$$\|\mathbf{T} - D_{\mathbf{y}}\mathbf{f}(\mathbf{y})\| = \|D_{\mathbf{y}}\mathbf{f}(\mathbf{y}^*) - D_{\mathbf{y}}\mathbf{f}(\mathbf{y})\| < \frac{1}{2\|\mathbf{T}^{-1}\|}$$

and so that $D_{\mathbf{y}}\mathbf{f}$ is invertible on $B_r(\mathbf{y}^*)$. The latter is possible because $D_{\mathbf{y}}\mathbf{f}(\mathbf{y})$ is continuous in \mathbf{y} , $D_{\mathbf{y}}\mathbf{f}(\mathbf{y}^*)$ is invertible, and the set of invertible matrices is open.

Take any $\mathbf{y}_1, \mathbf{y}_2 \in B_r(\mathbf{y}^*)$. By the Mean Value Theorem, there is $\bar{\mathbf{y}}$ in the line segment $\ell(\mathbf{y}_1, \mathbf{y}_2)$ with

$$F_{\mathbf{x}}(\mathbf{y}_1) - F_{\mathbf{x}}(\mathbf{y}_2) = [D_{\mathbf{y}}F_{\mathbf{x}}(\bar{\mathbf{y}})](\mathbf{y}_1 - \mathbf{y}_2)$$

For any $\mathbf{y} \in B_r(\mathbf{y}^*)$, including $\bar{\mathbf{y}}$,

$$D_{\mathbf{y}}F_{\mathbf{x}}(\mathbf{y}) = \mathbf{I} - \mathbf{T}^{-1}D_{\mathbf{y}}\mathbf{f}(\mathbf{y}) = \mathbf{T}^{-1}[\mathbf{T} - D_{\mathbf{y}}\mathbf{f}(\mathbf{y})]$$

Putting it together,

$$\begin{aligned} \|F_{\mathbf{x}}(\mathbf{y}_1) - F_{\mathbf{x}}(\mathbf{y}_2)\| &= \|[D_{\mathbf{y}}F_{\mathbf{x}}(\bar{\mathbf{y}})](\mathbf{y}_1 - \mathbf{y}_2)\| \\ &= \|\mathbf{T}^{-1}[\mathbf{T} - D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}})]\| \|\mathbf{y}_1 - \mathbf{y}_2\| \\ &\leq \|\mathbf{T}^{-1}\| \|\mathbf{T} - D_{\mathbf{y}}\mathbf{f}(\bar{\mathbf{y}})\| \|\mathbf{y}_1 - \mathbf{y}_2\| \\ &< \frac{1}{2} \|\mathbf{y}_1 - \mathbf{y}_2\|. \end{aligned}$$

Proof continues ...

15.34 Proof of the Inverse Function Theorem III

Remainder of Proof. We have shown that for each \mathbf{x} , $F_{\mathbf{x}}$ is a contraction on $B_r(\mathbf{y}^*)$. By the Contraction Mapping Theorem, it has a unique fixed point, which we call $\mathbf{y}_{\mathbf{x}}$.

The fixed point $\mathbf{y}_{\mathbf{x}}$ obeys

$$\begin{aligned}\mathbf{y}_{\mathbf{x}} &= F_{\mathbf{x}}(\mathbf{y}_{\mathbf{x}}) \\ &= \mathbf{y}_{\mathbf{x}} + \mathbf{T}^{-1}(f(\mathbf{y}_{\mathbf{x}}) - \mathbf{x}).\end{aligned}$$

This implies $f(\mathbf{y}_{\mathbf{x}}) = \mathbf{x}$. Define the function $\hat{\mathbf{y}}$ by $\hat{\mathbf{y}}(\mathbf{x}) = \mathbf{y}_{\mathbf{x}}$. Then $f(\hat{\mathbf{y}}(\mathbf{x})) = \mathbf{x}$, so $\hat{\mathbf{y}}$ is the inverse of f .

By Invariance of Domain, the inverse of f is continuous on $V = f(B_r(\mathbf{y}^*))$, which is open and contains \mathbf{x}^* . Finally, since f is \mathcal{C}^1 and $D\hat{\mathbf{y}} = (Df)^{-1}$ exists and is continuous, $\hat{\mathbf{y}} = f^{-1}$ is continuously differentiable on V . That is, $\hat{\mathbf{y}}$ is \mathcal{C}^1 on V with derivative given by equation (15.31.7). ■

15.35 Proof of Implicit Function Theorem I

With the Inverse Function Theorem in hand, we are ready to tackle the Implicit Function Theorem.

Proof (Implicit Function Theorem). We use the Inverse Function Theorem. The key is to define the proper mapping. Let¹³

$$\mathbf{G} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ \mathbf{g}(\mathbf{x}, \mathbf{y}) \end{pmatrix},$$

which maps from \mathbb{R}^{m+k} to \mathbb{R}^{m+k} . Here $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{g} \in \mathbb{R}^k$. Its derivative is

$$D\mathbf{G} = \begin{pmatrix} \mathbf{I}_m & \mathbf{0} \\ D_x\mathbf{g} & D_y\mathbf{g} \end{pmatrix},$$

which is invertible at $(\mathbf{x}^*, \mathbf{y}^*)$ because $(D_y\mathbf{g})(\mathbf{x}^*, \mathbf{y}^*)$ is invertible. In fact,

$$D\mathbf{G}^{-1} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -(D_y\mathbf{g})^{-1}D_x\mathbf{g} & \mathbf{I} \end{pmatrix}.$$

Proof continues ...

¹³ All of the arguments to \mathbf{g} , \mathbf{G} , and \mathbf{H} in this proof are vectors, not covectors. I've sometimes written them horizontally to save space.

15.36 Proof of Implicit Function Theorem II

Remainder of Proof. Applying the Inverse Function Theorem to \mathbf{G} , there is an $r > 0$ so that an inverse function \mathbf{H} is defined on $B_r(\mathbf{x}^*, \mathbf{y}^*)$ which maps to a neighborhood of $\mathbf{G}(\mathbf{x}^*, \mathbf{y}^*) = (\mathbf{x}^*, \mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)) = (\mathbf{x}^*, \mathbf{c}) \in \mathbb{R}^k$. We now write

$$\mathbf{H} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{H}_1(\mathbf{u}, \mathbf{v}) \\ \mathbf{H}_2(\mathbf{u}, \mathbf{v}) \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{G} \left(\mathbf{H} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \right) &= \begin{pmatrix} \mathbf{H}_1(\mathbf{u}, \mathbf{v}) \\ \mathbf{g}(\mathbf{H}_1(\mathbf{u}, \mathbf{v}), \mathbf{H}_2(\mathbf{u}, \mathbf{v})) \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}. \end{aligned}$$

Then $\mathbf{u} = \mathbf{H}_1(\mathbf{u}, \mathbf{v})$ and we can write

$$\mathbf{g}(\mathbf{u}, \mathbf{H}_2(\mathbf{u}, \mathbf{v})) = \mathbf{v}.$$

Setting $\mathbf{v} = \mathbf{c}$ and $\mathbf{u} = \mathbf{x}$, we obtain

$$\mathbf{c} = \mathbf{g}(\mathbf{x}, \mathbf{H}_2(\mathbf{x}, \mathbf{c})),$$

and define $\hat{\mathbf{y}}(\mathbf{x}) = \mathbf{H}_2(\mathbf{x}, \mathbf{c})$.

Then

$$\mathbf{g}(\mathbf{x}^*, \hat{\mathbf{y}}(\mathbf{x}^*)) = \mathbf{c} = \mathbf{g}(\mathbf{x}^*, \mathbf{y}^*),$$

showing that $\hat{\mathbf{y}}(\mathbf{x}^*) = \mathbf{y}^*$ since $D_{\mathbf{y}}\mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)$ is invertible. Finally, equation (15.31.7) follows by the Chain Rule. ■

15.37 Regular Manifolds

We can extend the concept of regularity to systems of equations and so to manifolds.

Regular Points. A point $(\mathbf{x}_0, \mathbf{y}_0) \in \mathbb{R}^{m+k}$ is a *regular point* of a \mathcal{C}^1 function $\mathbf{g} : \mathbb{R}^{m+k} \rightarrow \mathbb{R}^k$ if $\text{rank } D_{\mathbf{y}}\mathbf{g}(\mathbf{x}_0, \mathbf{y}_0) = k$.

We can use this to define regular manifolds.

Regular m -Manifolds. If every point of $M = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m+k} : \mathbf{g}(\mathbf{x}, \mathbf{y}) = \mathbf{c}\}$ is a regular point, we say that M is a *regular m -manifold*.

15.38 Manifolds as Solution Sets

We will show that every regular m -manifold has a \mathcal{C}^1 atlas and so is a \mathcal{C}^1 m -manifold.

Theorem 15.38.1. *Suppose $\mathbf{g}: \mathbb{R}^{m+k} \rightarrow \mathbb{R}^k$ is a \mathcal{C}^1 function and $M = \{\mathbf{z} \in \mathbb{R}^{m+k} : \mathbf{g}(\mathbf{z}) = \mathbf{c}\}$ is a regular m -manifold. Then for every point $\mathbf{z}^* \in M$ there is a relatively open set $U \subset M$ and a function $\varphi: U \rightarrow \mathbb{R}^m$ such that (U, φ) is a chart. These charts form a \mathcal{C}^1 atlas, making M a \mathcal{C}^1 manifold.*

Proof. Since $\text{rank } D\mathbf{g}(\mathbf{z}^*) = k$, we can divide the variables into two groups, $\mathbf{x} = (z_{i_1}, \dots, z_{i_m})$ and $\mathbf{y} = (z_{i_{m+1}}, \dots, z_{i_{m+k}})$ so that $D_{\mathbf{y}}\mathbf{g}(\mathbf{x}^*, \mathbf{y}^*)$ is invertible. The Implicit Function Theorem yields a \mathcal{C}^1 function $\hat{\mathbf{y}}: \mathbb{R}^m \rightarrow \mathbb{R}^k$ defined on ball $B_r(\mathbf{x}^*) \in \mathbb{R}^m$ with $r > 0$ such that $\mathbf{g}(\mathbf{x}, \hat{\mathbf{y}}(\mathbf{x})) = \mathbf{c}$. Set $U = B_r(\mathbf{x}^*, \mathbf{y}^*) \cap M$ and define $\varphi(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ for $\mathbf{x} \in U$.

I claim (U, φ) is a chart. Both φ and $\varphi^{-1} = (\mathbf{x}, \hat{\mathbf{y}}(\mathbf{x}))$ are \mathcal{C}^1 functions and φ is a homeomorphism between U and the open set $V = \varphi^{-1}(U) \subset M$. Thus (U, φ) is a chart. Since M is regular, we can generate a chart containing any point of M , showing that the charts cover M , forming an atlas.

All that is left is to show that the transition functions are \mathcal{C}^1 . Now suppose we have two charts (U, φ) and (V, ψ) where $U \cap V$ is non-empty. Now consider

$$\psi \circ \varphi^{-1}: \varphi(U \cap V) \rightarrow \psi(U \cap V).$$

Since both φ^{-1} and ψ are \mathcal{C}^1 , the Chain Rule tells us that the transition maps are \mathcal{C}^1 , showing that we have a \mathcal{C}^1 atlas. ■

15.39 Manifold or Not?

Graphs provide some of the simplest manifolds.

► **Example 15.39.1: An Atlas of One Chart.** Sometimes an atlas only needs one chart. Recall Example 15.25.1, where f is a real-valued function on \mathbb{R}^m and M is its graph. We defined a single chart, (M, π) where the projection is defined by $\pi(\mathbf{x}, y) = \mathbf{x}$. It has inverse $F(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}))$. Since the single chart covers M , it suffices to define an atlas for M . ◀

Circles require more than one chart. One way to see this is that any circle is compact, so its image is also compact, and cannot be an open set in \mathbb{R}^m . The charts we've used give us an atlas of 4 charts.

► **Example 15.39.2: An Atlas for every Circle.** Take the circle C about the origin with radius 5, $C \cap U_1 \cap U_3$ is the upper right quadrant of the circle and $\varphi_3^{-1}U_1 = \{y : 5 > y > 0\}$. Then

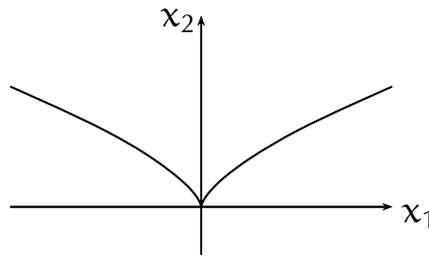
$$\varphi_1 \circ \varphi_3^{-1}(y) = \varphi_1(x_0 + \sqrt{25 - y^2}, y) = x_0 + \sqrt{25 - y^2}.$$

This is \mathcal{C}^1 on $\{y : 5 > y > 0\}$. Similarly, the other transition maps are \mathcal{C}^1 . This atlas makes C a \mathcal{C}^1 manifold. In fact, it makes C a \mathcal{C}^∞ manifold.

This same technique can be used to define an atlas for any circle. In fact the technique can also be used on spheres, although it requires more charts, $2(m + 1)$ for the unit m -sphere, which is $S^m = \{\mathbf{x} \in \mathbb{R}^{m+1} : \|\mathbf{x}\|_2 = 1\}$. ◀

15.40 Are these Manifolds?

In contrast, the curve with a cusp in Example 14.31.1 (shown below) is not a differentiable manifold as it is impossible to define a chart in a neighborhood of the origin that is compatible with a \mathcal{C}^1 atlas. This is due to the fact that the curve is not regular at the origin. Since the curve is homeomorphic to the real line, it is a topological manifold, but it is not a differentiable manifold.



As we saw with Bernoulli's lemniscate (section 15.8), curves that cross or have a T-intersection fail to even be topological manifolds.

► **Example 15.40.1: Regular Curves are Manifolds.** We saw how to construct charts for any regular curve in Example 15.26.2. Since this construction can be done at any point of C , the charts defined this way form an atlas for C . This shows that our original definition of a manifold is encompassed in the second definition using charts and an atlas. ◀

A similar procedure works for m -dimensional manifolds. It uses the Implicit Function Theorem.

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