20. Homogeneous and Homothetic Functions

Homogeneous and homothetic functions are closely related, but are used in different ways in economics. We focus on four main areas, starting with a look at homogeneity.

20.1 Homogeneous Functions

Homogeneity is related to the idea of returns to scale. Let's focus on constant returns to scale. A production function exhibits *constant returns to scale* if, whenever we increase all inputs in a given proportion, output changes by the same proportion.

In equations, let $f\colon \mathbb{R}^m_+ \to \mathbb{R}$ be a production function. It exhibits *constant returns to scale* if

$$f(t\mathbf{x}) = tf(\mathbf{x}) \tag{20.1.1}$$

for every $\mathbf{x} \in \mathbb{R}^m_+$ and $t > 0$.

Is my use of $t > 0$ rather than $t > 1$ bait and switch? No.

Theorem 20.1.1. *Suppose* $f(t\mathbf{x}) = tf(\mathbf{x})$ *for all* $t > 1$. *Then* $f(t\mathbf{x}) = tf(\mathbf{x})$ *for all* $t > 0$.

Proof. Equation 20.1.1 trivially holds for $t = 1$. Now suppose $0 < t < 1$. Then $1/t > 1$, so

$$f(\mathbf{x}) = f(t^{-1}t\mathbf{x}) = \frac{1}{t}f(t\mathbf{x})$$

implying $tf(\mathbf{x}) = f(t\mathbf{x})$ for $0 < t < 1$ ∎

Constant returns to scale functions are also called *homogeneous of degree one* because we multiply f by t to the first power.

20.1.1 Degrees of Homogeneity

Similarly, a function is homogeneous of degree γ if we multiply f by t raised to the γ power. More formally,

Homogeneous Function. On $\mathbb{R}^m_+$, a real-valued function is *homogeneous of degree* γ if

$$f(t\mathbf{x}) = t^\gamma f(\mathbf{x})$$

for every $\mathbf{x} \in \mathbb{R}^m_+$ and $t > 0$.

The degree of homogeneity need not be an integer. It can even be negative on $\mathbb{R}^m_{++}$.

Constant returns to scale functions are homogeneous of degree one. If $\gamma > 1$, homogeneous functions of degree γ have increasing returns to scale, and if $0 < \gamma < 1$, homogeneous functions of degree γ have decreasing returns to scale. However, many production functions with increasing or decreasing returns to scale are not homogeneous, such as $f(x) = x^2 + x^3$.

20.1.2 Cobb-Douglas Production and Returns to Scale

The Cobb-Douglas production function gives us examples of both increasing and decreasing returns to scale.

► **Example 20.1.2: Cobb-Douglas Production.** Cobb-Douglas production functions defined on $\mathbb{R}^2_+$ have the form

$$f(x, y) = Ax^{\alpha}y^{\beta}$$

with $A, \alpha, \beta > 0$. Here A is a productivity parameter.

Then

$$f(tx, ty) = A(tx)^{\alpha}(ty)^{\beta} = t^{\alpha+\beta}Ax^{\alpha}y^{\beta} = t^{\alpha+\beta}f(x, y)$$

showing that f is homogeneous of degree $\alpha + \beta$.

- If $\alpha + \beta < 1$, f has decreasing returns to scale.
- If $\alpha + \beta = 1$, f has constant returns to scale.
- If $\alpha + \beta > 1$, f has increasing returns to scale.

We can generalize this to $\mathbb{R}^m_+$. Cobb-Douglas functions on $\mathbb{R}^m_+$ have the form

$$f(\mathbf{x}) = A\prod_{i=1}^{m} x_i^{\gamma_i}$$

with each $\gamma_i > 0$ and $A > 0$. Such a function is homogeneous of degree $\sum_{i=1}^m \gamma_i$ on $\mathbb{R}^m_+$. Again, it exhibits constant returns to scale when $\sum_i \gamma_i = 1$, increasing returns when $\sum_i \gamma_i > 1$, and decreasing returns when $\sum_i \gamma_i < 1$. ◄

20.1.3 Examples of Homogeneous Functions

Homogeneous functions arise in both consumer's and producer's optimization problems. The cost, expenditure, and profit functions are homogeneous of degree one in prices. Indirect utility is homogeneous of degree zero in price-income pairs. Further, homogeneous production and utility functions are often used in empirical work.

On $\mathbb{R}^m_+$, the Leontief function $f(\mathbf{x}) = \min x_i$ is homogeneous of degree 1, as is the linear production function $f(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x}$.

We can easily make increasing or decreasing returns to scale functions by taking their exponentials. Thus the functions

$$(\min_i x_i)^\gamma \quad \text{and} \quad (\mathbf{a}\cdot\mathbf{x})^\gamma$$

are homogeneous of degree gamma.

The **constant elasticity of substitution** (CES) function on $\mathbb{R}^2_+$ is

$$f(\mathbf{x}) = [\delta x_1^\rho + (1-\delta)x_2^\rho]^{\nu/\rho}$$

for $\nu > 0$ and $\rho < 1$, $\rho \neq 0$ is homogeneous of degree ν. Here $\sigma = (1-\rho)^{-1}$ is the **elasticity of substitution**.

20.1.4 More Examples of Homogeneous Functions

Any monomial,

$$a \prod_{i=1}^{m} x_i^{\gamma_i}$$

is homogeneous of degree $\gamma = \sum_{i=1}^{m} \gamma_i$ on $\mathbb{R}_+^m$. A sum of monomials of degree γ is homogeneous of degree γ, the sum of monomials of differing degrees is not homogeneous.

Additional examples of homogeneous functions include:

$$x^{3/2} + 3x^{1/2}y, \quad \sqrt{x+y+z}, \quad Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$$
$$x^8 + 5x^4y^4 + 10x^3y^5 + 7xy^7 + 13y^8.$$

20.1.5 Some Inhomogeneous Functions

The following functions are not homogeneous:

$$\sin xyz, \quad x^2 + y^3, \quad 3xyz + 3x^2z - 3xy$$
$$\frac{xy + yz + xz}{x + z^2}, \quad \exp(x^2 + y^2).$$

Most quasi-linear utility functions, such as $u(\mathbf{x}) = x_1 + x_2^{1/2}$ are not homogeneous of any degree. The linear term means that they can only be homogeneous of degree one, meaning that the function can only be homogeneous if the non-linear term is also homogeneous of degree one. Non-linear cases that **are** homogeneous of degree one require at least two goods. One example is

$$x_1 + (x_1x_2)^{1/2}$$

which is homogeneous of degree one on $\mathbb{R}^2_+$.

20.1.6 Cones: The Natural Setting for Homogeneity

Homogeneous functions require us to know $f(t\mathbf{x})$ when we know $f(\mathbf{x})$. That means that $t\mathbf{x} \in \operatorname{dom} f$ whenever $\mathbf{x} \in \operatorname{dom} f$. Sets with that property are called cones. Cones are the natural domain of homogeneous functions.

Cone. Let V be a vector space. A set $C \subset V$ is a *cone* if for every $\mathbf{x} \in C$ and $t > 0$, $t\mathbf{x} \in C$. Equivalently, C is a *cone* if $tC \subset C$ for all $t > 0$.

Cones include the positive orthant $\mathbb{R}^m_+$, the strictly positive orthant $\mathbb{R}^m_{++}$, any vector subspace of $\mathbb{R}^m$, any ray in $\mathbb{R}^m$ and any set of non-negative linear combinations of a collection of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_i\}$. The set $\{(x, y) : x, y \geq 0, y \leq x\}$ is an example of the last as it can also be written $\{\mathbf{x} = t_1(1, 0) + t_2(1, 1) : t_1, t_2 \geq 0\}$.

Cones can also be spiky. The set

$$\left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = t(1, 2) \text{ or } \mathbf{x} = t(1, 1) \text{ or } \mathbf{x} = t(3, 1) \text{ for } t \geq 0\right\}$$

is also a cone, even though it consists of three unrelated rays from the origin, as in Figure 20.1.3.

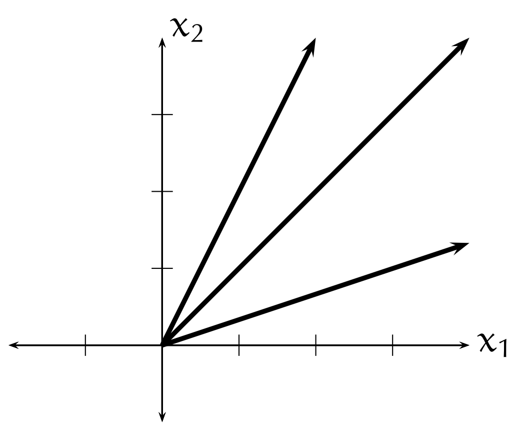

Figure 20.1.3: A spiky cone, made up by the union of three rays through the origin.

20.1.7 Homogeneous Functions on Cones

We can redefine homogeneity to apply to functions defined on any cone.

Homogeneous Function. Let C be a cone in a vector space V. A function $f\colon C \to \mathbb{R}$ is *homogeneous of degree* γ if

$$f(t\mathbf{x}) = t^{\gamma} f(\mathbf{x})$$

for every $\mathbf{x} \in \mathbb{R}^m$ and $t > 0$.

Restricting the domain of a homogeneous function so that it is not all of $\mathbb{R}^m_+$ allows us to expand the notation of homogeneous functions to negative degrees by avoiding division by zero. We can include functions such as $1/\|\mathbf{x}\|_2$, which is homogeneous of degree -1 on the cone $\mathbb{R}^m_{++}$.

Restricting the domain also allows us to consider quotients such as $f(\mathbf{x})/g(\mathbf{x})$ where f is homogeneous of degree γ_1 and g is homogeneous of degree γ_2, both on $\mathbb{R}^m_{++}$. The quotient is homogeneous of degree $\gamma_1 - \gamma_2$ on $\mathbb{R}^m_{++}$. To see that, consider

$$\frac{f(t\mathbf{x})}{g(t\mathbf{x})} = \frac{t^{\gamma_1} f(\mathbf{x})}{t^{\gamma_2} g(\mathbf{x})} = t^{\gamma_1 - \gamma_2} \frac{f(\mathbf{x})}{g(\mathbf{x})}$$

which shows the quotient is homogeneous of degree $\gamma_1 - \gamma_2$.

20.2 Properties of Homogeneous Functions

Homogeneous functions have some special properties. For example, their derivatives are homogeneous, the slopes of level sets are constant along rays through the origin, and you can easily recover the original function from the derivative (Euler's Theorem). The latter has implications for firms' profits.

20.2.1 Derivatives of Homogeneous Functions

When a function is homogeneous, its derivative is also homogeneous, but with the degree reduced by one.

Theorem 20.2.1. *Let $C \subset \mathbb{R}^m$ be an open cone and $f\colon C \to \mathbb{R}$ is $\mathcal{C}^1$ and homogeneous of degree γ. Then the derivative Df is homogeneous of degree $(\gamma - 1)$.*

Proof. By homogeneity, $f(t\mathbf{x}) = t^\gamma f(\mathbf{x})$. Taking the $\mathbf{x}$ derivative of that equation, we obtain

$$t\,Df(t\mathbf{x}) = t^\gamma\,Df(\mathbf{x}).$$

Dividing by t, we find that

$$Df(t\mathbf{x}) = t^{(\gamma-1)}\,Df(\mathbf{x}),$$

showing that Df is homogeneous of degree $\gamma - 1$. ∎

In Theorem 20.2.1, we assumed the cone C was open so that Df could be defined at all points of C.

20.2.2 Anti-Derivatives of Homogeneous Functions

The converse fails. If Df is homogeneous of degree β, we cannot conclude that f is homogeneous of degree $(\beta + 1)$. For example, let $m = 2$ and consider $f(\mathbf{x}) = 1 + x_1 x_2$, which is not homogeneous of any degree.

A quick calculation shows that $(D_{\mathbf{x}} f)(\mathbf{x}) = (x_2, x_1)$ which is homogeneous of degree one, even though f is not homogeneous. Fortunately, addition of a constant is the main thing that goes wrong with the converse when Df is homogeneous of degree β for $\beta \neq -1$.

Functions with $\beta = -1$ can suffer from two other types of complications.

The first complication involves logarithmic functions. Suppose $f(\mathbf{x}) = b \ln \phi(\mathbf{x})$ where ϕ is homogeneous of degree one with $\phi > 0$. Then

$$Df = \frac{b}{\phi(\mathbf{x})} D\phi(\mathbf{x}),$$

which is homogeneous of degree minus one ($\beta = -1$).

The second type of complication occurs when there is more than one variable. In $\mathbb{R}^m$ with $m > 1$, functions can be homogeneous of degree zero without being constant.

One such function is $g(\mathbf{x}) = x_1/(x_1 + x_2)$. Its derivative is

$$Dg = \frac{1}{(x_1 + x_2)^2} (x_2, -x_1),$$

which is clearly homogeneous of degree minus one.

20.2.3 Converse to Theorem 20.2.1

We state a near converse to Theorem 20.2.1 without proof. A proof using Stokes' Theorem can be found in my micro theory manuscript.

Theorem 20.2.2. *Suppose* $f \in \mathcal{C}^2$ *on* $\mathbb{R}^L_{++}$ *and* Df *is homogeneous of degree* β *in* $\mathbf{x}$ *with* $Df \neq 0$ *on* $\mathbb{R}^L_{++}$.

1. *If* $\beta \neq -1$, *there is a constant* c *and a function* $\nu(\mathbf{x})$ *that is homogeneous of degree one such that* $f(\mathbf{x}) = c + \big(\nu(\mathbf{x})\big)^{1+\beta}$.
2. *If* $\beta = -1$, *there is a constant* b, *a function* $\phi(\mathbf{x})$ *that is homogeneous of degree zero, and another function* $\nu(\mathbf{x})$ *that is homogeneous of degree one with* $f(\mathbf{x}) = \phi(\mathbf{x}) + b \ln \nu(\mathbf{x})$. *Either* b *or* ϕ *may be zero.*

20.2.4 Homogeneity and Marginal Rates of Substitution

The fact that derivatives of homogeneous functions are also homogeneous has consequences for the shape of indifference curves and isoquants. It implies that the marginal rate of substitution and marginal rate of technical substitution are constant along rays through the origin. That is, they are homogeneous of degree zero.

Theorem 20.2.3. *Suppose* $f\colon \mathbb{R}^L_{++} \to \mathbb{R}$ *is homogeneous of degree* γ *and differentiable. If* $\text{MRS}_{k\ell}(\mathbf{x})$ *exists, then* $\text{MRS}_{k\ell}(\mathbf{x}) = \text{MRS}_{k\ell}(t\mathbf{x})$ *for all* $t > 0$ *and* $\mathbf{x} \in \mathbb{R}^L_{++}$.

Proof. To see this let f be homogeneous of degree γ and consider $\text{MRS}_{ij} = (\partial f/\partial x_i)\big/(\partial f/\partial x_j)$. We calculate

$$\text{MRS}_{ij}(t\mathbf{x}) = \frac{\dfrac{\partial f}{\partial x_i}(t\mathbf{x})}{\dfrac{\partial f}{\partial x_j}(t\mathbf{x})} = \frac{t^{\gamma-1}\dfrac{\partial f}{\partial x_i}(\mathbf{x})}{t^{\gamma-1}\dfrac{\partial f}{\partial x_j}(\mathbf{x})} = \frac{\dfrac{\partial f}{\partial x_i}(\mathbf{x})}{\dfrac{\partial f}{\partial x_j}(\mathbf{x})} = \text{MRS}_{ij}(\mathbf{x}).$$

∎

20.2.5 Homogeneity and Indifference Curves

The fact that marginal rates of substitution are constant along rays has consequences. It implies that income expansion paths and scale expansion paths are rays through the origin whenever the utility or production function is homogeneous.

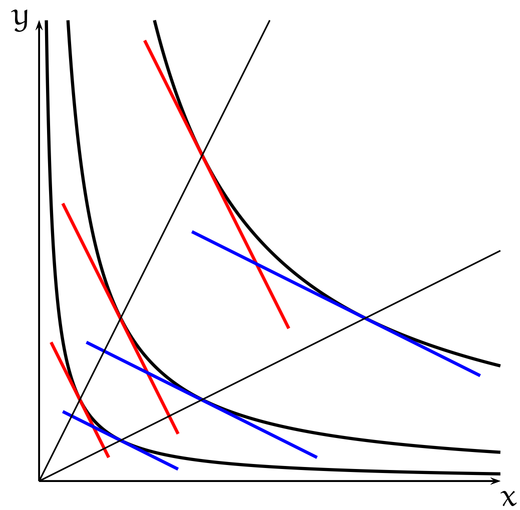

Figure 20.2.4: Here are three indifference curves for the Cobb-Douglas utility function $u(x, y) = \sqrt{xy}$. I've also draw two rays from the origin. Notice how the slope of the indifference curves remains the same along each ray. The red tangent lines all have slope -2, while the blue tangents have slope -0.5.

20.2.6 Euler's Theorem

The second important property of homogeneous functions is given by Euler's Theorem.

Euler's Theorem. *Let* $f\colon \mathbb{R}^m_{++} \to \mathbb{R}$ *be* $\mathcal{C}^1$. *Then* f *is homogeneous of degree* γ *if and only if* $\big[D_{\mathbf{x}} f(\mathbf{x})\big]\mathbf{x} = \gamma f(\mathbf{x})$, *that is*

$$\sum_{i=1}^{m} x_i \frac{\partial f}{\partial x_i}(\mathbf{x}) = \gamma f(\mathbf{x}).$$

Proof. Define $\varphi(t) = f(t\mathbf{x})$. By the chain rule, $d\varphi/dt = \big[Df(t\mathbf{x})\big]\mathbf{x}$.

(Only If) If f is homogeneous of degree γ, we can also write $\varphi(t) = t^\gamma f(\mathbf{x})$ so that $d\varphi/dt = \gamma t^{\gamma-1} f(\mathbf{x})$. Setting $t = 1$ we obtain $(Df)\mathbf{x} = \gamma f(\mathbf{x})$.

(If) Conversely, when $\big[D_{\mathbf{x}} f(\mathbf{x})\big]\mathbf{x} = \gamma f(\mathbf{x})$, we have

$$\frac{d\varphi}{dt} = \big[D_{\mathbf{x}} f(t\mathbf{x})\big]\mathbf{x} = \frac{1}{t}\big[D_{\mathbf{x}} f(t\mathbf{x})\big](t\mathbf{x}) = \frac{\gamma f(t\mathbf{x})}{t} = \frac{\gamma\varphi(t)}{t}.$$

It follows that

$$\frac{\gamma}{t} = \frac{1}{\varphi}\frac{d\varphi}{dt} = \frac{d(\ln\varphi)}{dt}.$$

Integrating from $t = 1$ to t, we obtain $\ln\varphi(t) - \ln\varphi(1) = \gamma \ln t$. Taking the exponential yields $\varphi(t) = \varphi(1)t^\gamma$. Recalling that $\varphi(1) = f(\mathbf{x})$, we rewrite our result as $f(t\mathbf{x}) = t^\gamma f(\mathbf{x})$. The function f is homogeneous of degree γ. ∎

20.2.7 Gradients and Euler's Theorem

Euler's Theorem is sometimes written using the gradient, $\nabla f = Df^T$, which is a column vector. In that case, the condition can be written

$$\boldsymbol{x} \cdot \nabla_x f = \gamma f(\boldsymbol{x}).$$

Either way, it means that

$$\sum_{i=1}^{m} x_i \frac{\partial f}{\partial x_i} = \gamma f(\boldsymbol{x}).$$

20.2.8 Wicksteed's Theorem I

Euler's Theorem has an important application in the theory of a competitive firms.

Suppose f is a homogeneous production function of degree γ. If the firm is a price-taker in both input and output markets, profits are $pf(\mathbf{x}) - \boldsymbol{w}\cdot\mathbf{x}$ where p is the output price and $\boldsymbol{w}$ the vector of factor prices.

To maximize profit we set the derivative of profit equal to zero.

$$p\,Df(\mathbf{x}^*) = \boldsymbol{w}^T.$$

This tells us that for each good i, the value of marginal revenue must be equal to the factor price:

$$p\frac{\partial f}{\partial x_i} = w_i.$$

What about profits? Euler's Theorem provides the answer. By Euler's Theorem,

$$\gamma f(\mathbf{x}) = \sum_i x_i \frac{\partial f}{\partial x_i}.$$

We multiply by p to obtain

$$\gamma p f(\mathbf{x}) = \sum_i x_i p \frac{\partial f}{\partial x_i} = \sum_i x_i w_i = \boldsymbol{w}\cdot\mathbf{x}$$

by the first order conditions.

20.2.9 Wicksteed's Theorem II

The last equation can be written as

$$pf(\mathbf{x}) = \frac{1}{\gamma}\mathbf{w}\cdot\mathbf{x}.$$

In other words, revenue is $1/\gamma$ times cost.

When $\gamma = 1$, this shows that revenue equals cost. Profit is zero. This is known as **Wicksteed's Theorem**.[1]

When $\gamma < 1$ (decreasing returns to scale), revenues are greater than cost. Such a firm is profitable.

When $\gamma > 1$ (increasing returns to scale), revenues are less than cost. The firm is making a loss. This means that in the long run, no perfectly competitive firm can have $\gamma > 1$. If they did, they would be driven out of business. This shows that increasing returns are incompatible with perfect competition.

[1] This was first noted by Wicksteed in Philip H. Wicksteed (1894), *Essay on the Co-ordination of the Laws of Distribution*, Macmillan, London., but his proof was incorrect. The first correct proof is that of Flux in Alfred W. Flux (1894), Review of Wicksteed's *Essay on the Laws of the Coordination of Distribution*, *Econ. J.* **4**, 308–313.

20.2.10 Monotonicity

We define three types of monotonicity for real-valued functions of m variables, based on the three types of inequality.[2]

Monotonicity. A real-valued function f defined on a subset of $\mathbb{R}^m$ is *monotonic* if $f(\mathbf{x}) \geq f(\mathbf{y})$ whenever $\mathbf{x} \geq \mathbf{y}$. It is *strongly monotonic* if $f(\mathbf{x}) > f(\mathbf{y})$ whenever $\mathbf{x} > \mathbf{y}$. Finally, it is *strictly monotonic* if $f(\mathbf{x}) > f(\mathbf{y})$ whenever $\mathbf{x} \gg \mathbf{y}$.

For functions on $\mathbb{R}$, strong and strict monotonicity are the same. However, they differ on $\mathbb{R}^m$ when $m > 1$.

It is easy to see that strongly monotonic functions are both strictly monotonic and monotonic.

Strictly monotonic functions on $\mathbb{R}^m_+$ that are continuous are also monotonic functions: If $\mathbf{y} \geq \mathbf{x}$, define

$$\mathbf{y}^n = \frac{1}{n}\mathbf{e} + \mathbf{y} \gg \mathbf{y} \geq \mathbf{x}.$$

Then $f(\mathbf{y}^n) > f(\mathbf{x})$. Taking the limit shows $f(\mathbf{y}) \geq f(\mathbf{x})$.

[2] Recall that $\mathbf{x} \geq \mathbf{y}$ means $x_i \geq y_i$ for all i; $\mathbf{x} > \mathbf{y}$ means $\mathbf{x} \geq \mathbf{y}$, but not $\mathbf{x} = \mathbf{y}$; $\mathbf{x} \gg \mathbf{y}$ means $x_i > y_i$ for all i. Thus $\mathbf{x} \gg \mathbf{y}$ implies $\mathbf{x} > \mathbf{y}$ and $\mathbf{x} > \mathbf{y}$ implies $\mathbf{x} \geq \mathbf{y}$.

20.2.11 Strictly Monotonic but not Monotonic

However, there may be problems doing this in other subsets of $\mathbb{R}^m$. Sometimes, strictly monotonic functions are not monotonic!

Consider the subset S given by

$$S = \{(x, y) \in \mathbb{R}^2_+ : x = y \text{ or } x = 0 \text{ or } y = 0\}.$$

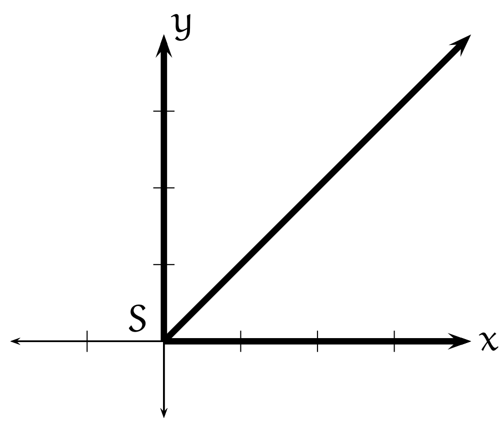

Figure 20.2.5: The set S.

On S, define $f(x, y) = x + y$ when $x = y$, $f(x, y) = -x - y$ otherwise. This is strictly monotonic. No point with $x = 0$ or $y = 0$ can be strictly greater than any other point in S. To do so both coordinates must be strictly positive, which cannot happen on either axis.

When it comes to strict monotonicity, we only have to ask if $\mathbf{x} \gg \mathbf{y}$ when $\mathbf{x}$ is on the 45° line. Then $\mathbf{x} = (x, x)$ dominates points of the form (y, y) with $y < x$, $(0, y)$ with $y < x$ and $(y, 0)$ with $y < x$. In each case, f is smaller on $\mathbf{y}$ than on $\mathbf{x}$, establishing strict monotonicity.

Amazingly, the function f is strictly monotonic even though it is strictly decreasing on both the horizontal and vertical axes.

20.2.12 Homogeneity and Monotonicity

We saw earlier that the homogeneous of degree zero function $g(\mathbf{x}) = x_1/(x_1 + x_2)$ is not monotonic. This is normal for such functions. Indeed, we can use Euler's Theorem to show that homogeneous of degree zero functions **cannot be monotonic** when there are two or more variables.

Theorem 20.2.6. *Any function* $f \in \mathcal{C}^1(\mathbb{R}^m_{++})$ *for* $m > 1$ *that is homogeneous of degree zero is not monotonic (meaning* $Df \not\gg \mathbf{0}$*).*

Proof. We prove this by applying Euler's Theorem,

$$0 = \sum_i x_i \frac{\partial f}{\partial x_i}.$$

If every $\partial f/\partial x_i > 0$, the right-hand side is positive. This is impossible as it must be zero, so there has to be j with $\partial f/\partial x_j < 0$,. The function f cannot be monotonic. ∎

However, it is possible to combine a logarithmic form with a homogeneous of degree zero form to get an increasing function with a derivative that is homogeneous of degree minus one. The function

$$f(x_1, x_2) = \frac{x_1}{x_1 + x_2} + \ln(x_1 + x_2)$$

is such a case. Verifying that f is increasing is left as an exercise to the reader.

20.3 Cardinality and Ordinality

The terms *cardinal* and *ordinal* are often used when discussing utility functions. Cardinality and ordinality are not properties of the functions themselves. They describe how we **interpret** the functions. When we say a production function is cardinal, we mean that the output numbers have direct meaning. Two tons of steel and three tons of steel are different things.

In contrast, the absolute level of utility doesn't mean anything by itself. What is important are comparisons of utility levels—which consumption bundles yield higher utility levels and which yield less utility.

Production functions are cardinal because the actual quantity produced matters. Utility functions are ordinal because only relative utility rankings have meaning.

20.3.1 Ordinally Equivalent Functions

For utility functions, what matters is not the level of $u(\mathbf{x})$ or $u(\mathbf{y})$, but whether we prefer bundle $\mathbf{x}$ or bundle $\mathbf{y}$. That is, whether $u(\mathbf{x}) \geq u(\mathbf{y})$ or $u(\mathbf{y}) \geq u(\mathbf{x})$.

We say that two functions $u, v\colon \mathfrak{X} \to \mathbb{R}$ are *ordinally equivalent* if $u(\mathbf{x}) \geq u(\mathbf{y})$ if and only if $v(\mathbf{x}) \geq v(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$.

Ordinally equivalent utility functions describe the same preference order.

Theorem 20.3.1. *Ordinal equivalence between functions from $\mathfrak{X} \to \mathbb{R}$ is an equivalence relation.*

Proof. The relation is reflexive. A function $u\colon \mathfrak{X} \to \mathbb{R}$ is ordinally equivalent to itself because $u(\mathbf{x}) \geq u(\mathbf{y})$ if and only if $u(\mathbf{x}) \geq u(\mathbf{y})$. Yes, that was pretty trivial.

The relation is symmetric. If u is ordinally equivalent to v, then v is trivially ordinally equivalent to u. Here $u(\mathbf{x}) \geq u(\mathbf{y})$ if and only if $v(\mathbf{x}) \geq v(\mathbf{y})$, so $v(\mathbf{x}) \geq v(\mathbf{y})$ if and only if $u(\mathbf{x}) \geq u(\mathbf{y})$.

Finally, the relation is transitive. This is barely harder to show. If u is ordinally equivalent to v and v is ordinally equivalent to a function w then $u(\mathbf{x}) \geq u(\mathbf{y})$ if and only if $v(\mathbf{x}) \geq v(\mathbf{y})$ if and only if $w(\mathbf{x}) \geq w(\mathbf{y})$, so $u(\mathbf{x}) \geq u(\mathbf{y})$ if and only if $w(\mathbf{x}) \geq w(\mathbf{y})$, showing that u is ordinally equivalent to w. This proves transitivity. ∎

Ordinal. A property of a function $f\colon \mathfrak{X} \to \mathbb{R}$ is *ordinal* if it shared by all ordinally equivalent functions.

20.3.2 Two Types of Equivalence

Warning! Our treatment of equivalence is slightly different from that in Simon and Blume (section 20.3) defined equivalence by an equality rather than an inequality (they focus on the set of indifference curves).

The reason we use an inequality is that if we used equality, as they do, the functions $u(x, y) = x + y$ and $v(x, y) = -x - y$ would be equivalent functions. However, as utility functions, they are completely different. For u, more is better, for v, more is worse. Our definition maintains the sense of utility, by maintaining the orientation of the indifference map.

If two functions are equivalent in our sense, they are equivalent in the Simon and Blume sense. As the above example shows, the converse is false.

Theorem 20.3.2. *If u and v are equivalent functions on $\mathfrak{X}$, then $u(\mathbf{x}) = u(\mathbf{y})$ if and only if $v(\mathbf{x}) = v(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$.*

Proof. Now $u(\mathbf{x}) = u(\mathbf{y})$ if and only if both $u(\mathbf{x}) \geq u(\mathbf{y})$ and $u(\mathbf{y}) \geq u(\mathbf{x})$. By equivalence, that holds if and only if both $v(\mathbf{x}) \geq v(\mathbf{y})$ and $v(\mathbf{y}) \geq v(\mathbf{x})$. The last pair of statements hold if and only if $v(\mathbf{x}) = v(\mathbf{y})$. ∎

Corollary 20.3.3 immediately follows.

Corollary 20.3.3. *If u and v are ordinally equivalent functions on a set $\mathfrak{X}$, then*

$$\begin{aligned} u(\mathbf{x}) > u(\mathbf{y}) \quad &\textit{if and only if} \quad v(\mathbf{x}) > v(\mathbf{y}) \\ u(\mathbf{x}) = u(\mathbf{y}) \quad &\textit{if and only if} \quad v(\mathbf{x}) = v(\mathbf{y}) \\ u(\mathbf{x}) < u(\mathbf{y}) \quad &\textit{if and only if} \quad v(\mathbf{x}) < v(\mathbf{y}) \end{aligned}$$

20.3.3 Optima of Ordinally Equivalent Functions

When we maximize or minimize ordinally equivalent function, the maxima or minima occur at the same point or points.

Theorem 20.3.4. *Suppose u and v are ordinally equivalent functions defined on a set S. Let S^* be the set of maximizers (minimizers) of u over S. Then S^* is also the set of maximizers (minimizers) of v over S.*

Proof. Now $\mathbf{x}^* \in S$ maximizes u over S if and only $u(\mathbf{x}^*) \geq u(\mathbf{x})$ for all $\mathbf{x} \in S$. By ordinal equivalence, this happens if and only if $v(\mathbf{x}^*) \geq v(\mathbf{x})$ for all $\mathbf{x} \in S$. ∎

If $\mathbf{x}^*$ is a strict maximizer or minimizer, S^* consists of a single point, so strict maximization and minimization are included in our result.

Interestingly enough, Theorem 20.3.4 applies even if u are v are considered cardinal functions. You can apply this to production functions.

It is sometimes useful to transform functions into an ordinally equivalent form before maximizing them. For example, the Cobb-Douglas function

$$u(\mathbf{x}) = A \prod_{i=1}^{m} x_i^{\gamma_i}$$

is equivalent to

$$v(\mathbf{x}) = \ln A + \sum_{i=1}^{m} \gamma_i \ln x_i,$$

where $\partial v/\partial x_i = \gamma_i/x_i$ depends only on x_i, unlike $\partial u/\partial x_i$.

20.3.4 Monotonic Transformations

Monotonic transformations will allow us to characterize all ordinally equivalent functions.

Monotonic Transformation. Let I be an interval in the real line. A function $\phi\colon I \to \mathbb{R}$ is a *monotonic transformation of* I if ϕ is strictly increasing on I. Moreover, if f is a real-valued function of m variables and ϕ is a monotonic transformation on its image, we say

$$\phi \circ f\colon \mathbf{x} \mapsto \phi\big(f(\mathbf{x})\big)$$

is a monotonic transformation of f.

If ϕ is differentiable with $\phi' > 0$, then ϕ is a monotonic transformation. Consideration of $\phi(x) = x^3$ on $I = (-1, 1)$ shows that $\phi' > 0$ is sufficient, but not necessary for a monotonic transformation.

NB: Note that monotonicity is defined differently for functions and transformations. Monotonic transformations are strictly increasing.

20.3.5 Monotonic Transformations and Ordinal Equivalence

One important result is that two functions are ordinally equivalent if and only if there are monotonic transformations of each into the other.

Theorem 20.3.5. *Let $u, v \colon \mathfrak{X} \to \mathbb{R}$. Then u and v are equivalent utility functions on $\mathfrak{X}$ if and only if there is a monotonic transformation ϕ with $u = \phi \circ v$.*

Proof. **If**: Suppose $u = \phi \circ v$. Because ϕ is strictly increasing, $v(\mathbf{x}) \geq v(\mathbf{y})$ if and only if $u(\mathbf{x}) = \phi\big(v(\mathbf{x})\big) \geq \phi\big(v(\mathbf{y})\big) = u(\mathbf{y})$, showing that the two functions are equivalent.

Only If: Suppose u and v are equivalent. Let $\bar{u} \in \operatorname{ran} u$. There is $\mathbf{x} \in \mathfrak{X}$ with $\bar{u} = u(\mathbf{x})$. Define $\phi(\bar{u}) = v(\mathbf{x})$.

The function v is well-defined.[3] Had we picked any other $\mathbf{y}$ with $u(\mathbf{y}) = \bar{u}$, we would have $v(\mathbf{x}) = v(\mathbf{y})$ by Theorem 20.3.2. This shows that the definition of $\phi(\bar{u}) = v(\mathbf{x})$ depends only on $\bar{u}$, not on our choice of $\mathbf{x}$ with $u(\mathbf{x}) = \bar{u}$.

Now $v(\mathbf{x}) = \phi\big(u(\mathbf{x})\big)$ for all $\mathbf{x} \in \mathfrak{X}$. We need only show that ϕ is increasing. Suppose $\bar{u}_0 < \bar{u}_1$. Take $\mathbf{x}_i$ with $u(\mathbf{x}_i) = \bar{u}_i$, $i = 0, 1$. Then $u(\mathbf{x}_1) > u(\mathbf{x}_0)$. By Corollary 20.3.3, $v(\mathbf{x}_1) > v(\mathbf{x}_0)$. Then

$$\phi(\bar{u}_1) = v(\mathbf{x}_1) > v(\mathbf{x}_0) = \phi(\bar{u}_0).$$

This shows that ϕ is increasing and completes the proof. ∎

[3] In mathematics, "well-defined" means that the definition is unambiguous.

20.4 Homothetic Functions

Homotheticity is a generalization of homogeneity. In fact, we will show that homothetic functions are the ordinal equivalent of homogeneous functions.

Homothetic Function. Let C be a cone. A function $f\colon C \to \mathbb{R}$ is *homothetic* if for every $\mathbf{x}, \mathbf{y} \in C$ and $t > 0$, $f(\mathbf{x}) \geq f(\mathbf{y})$ if and only if $f(t\mathbf{x}) \geq f(t\mathbf{y})$.

One consequence of the definition of homotheticity is that f is equivalent to g defined by $g(\mathbf{x}) = f(t_0\mathbf{x})$.

By the definitions, any homogeneous function is homothetic.

Theorem 20.4.1. *If u is homothetic and v is ordinally equivalent to u, then v is also homothetic.*

Proof. Suppose $v(\mathbf{x}) \geq v(\mathbf{y})$. Then $u(\mathbf{x}) \geq u(\mathbf{y})$ by equivalence. For all $t > 0$, $u(t\mathbf{x}) \geq u(t\mathbf{y})$ by homotheticity of u, and then $v(t\mathbf{x}) \geq v(t\mathbf{y})$ for all $t > 0$ by equivalence. ∎

20.4.1 More on Homotheticity

For a utility function, homotheticity means that preferences are invariant under scalar multiplication in the sense that the set of indifference curves is unchanged when all consumption bundles are multiplied by the same positive number. More precisely, preferences are invariant under homothetic transformations centered on the origin.

Homothetic preferences include commonly used functional forms such as Cobb-Douglas utility and constant elasticity of substitution utility.

Homogeneous functions are a special type of homothetic function. The Homothetic Representation Theorem will show that monotonic continuous and homothetic function is a monotonic transformation of a homogeneous function of degree one.

Homotheticity in economics is based on comparing positive scalar multiples of vectors. By restricting our attention to consumption sets that are cones, we ensure that scalar multiplication is always possible. Such scaling preserves the shapes of objects, including indifference surfaces. It only changes their scale.

20.4.2 Marginal Rates of Substitution and Equivalence

By Theorem 20.3.5, the functions x, x^2, x^7, and e^x are all monotonic transformations of $f(x) = x$, and so all equivalent. If we restrict our attention to $\mathbb{R}_{++}$, $\ln x$ also qualifies. Similarly, xy, $xy + (xy)^4$, $\log xy$, and $(xy)^{1/2}$ are all equivalent on $\mathbb{R}^2_+$.

When utility is differential, one property preserved by monotonic differentiable transformations is the marginal rate of substitution. Suppose $v(\mathbf{x}) = \phi\big(u(\mathbf{x})\big)$ and $\phi' > 0$. Then by the Chain Rule,

$$\mathrm{MRS}^v_{ij} = \frac{\dfrac{\partial v}{\partial x_i}}{\dfrac{\partial v}{\partial x_j}} = \frac{\phi' \dfrac{\partial u}{\partial x_i}}{\phi' \dfrac{\partial u}{\partial x_j}} = \frac{\dfrac{\partial u}{\partial x_i}}{\dfrac{\partial u}{\partial x_j}} = \mathrm{MRS}^u_{ij}$$

provided the marginal rate of substitution makes sense (i.e., $\frac{\partial v}{\partial x_j} \neq 0$). Since $\phi' > 0$, the monotonic transformation doesn't affect this.

20.4.3 MRS Homogeneity implies Homotheticity

There is a converse, which we will state, but not prove.

Theorem 20.4.2. *Suppose* $f\colon \mathbb{R}^m_{++} \to \mathbb{R}$ *is* $\mathcal{C}^2$, $Df \gg \mathbf{0}$, *and* $MRS_{k\ell}$ *is homogeneous of degree zero in* $\mathbf{x}$ *for every* k *and* ℓ. *Then* f *is homothetic.*

Proof. A proof may be found in Lau.[4] Lau uses the slightly weaker assumption that there is some j with $\partial f/\partial x_j \neq 0$, in which case the MRS condition must be restated to work around the fact that $MRS_{k\ell}$ may not be defined for all pairs k and ℓ. Lau does not do this, but the replacement for the MRS condition is that for all k and ℓ, there are homogeneous of degree zero functions $g_{k\ell}$ such that $\partial f/\partial x_k = g_{k\ell} \times (\partial f/\partial x_\ell)$. When $Df \gg \mathbf{0}$, this is equivalent to the marginal rates of substitution being homogeneous of degree zero in $\mathbf{x}$. ∎

[4] Lemma 1 in Lawrence J. Lau (1969) Duality and the structure of utility functions *J. Econ. Theory*, **1**, 374–396.

20.4.4 A Key Lemma

12/01/22

One remarkable fact is that every continuous homothetic function is a monotonic transformation of a homogeneous function.

The key part of the theorem is separated as a lemma, for it is of more general use. In particular, with some minor modifications, it can be used to show that any continuous and monotonic preference order on $\mathbb{R}^m_+$ can be represented by a continuous utility function.[5]

Lemma 20.4.3. *Suppose $\mathbf{x}^* \in \mathbb{R}^m_{++}$ and f is continuous and increasing on $\mathbb{R}^m_+$. If $\mathbf{x} \in \mathbb{R}^m_+$, there is a unique $\alpha \geq 0$ with $f(\alpha\mathbf{x}^*) = f(\mathbf{x})$.*

Proof. Let $\mathbf{x} \in \mathbb{R}^m_+$. Choose $\bar{\alpha}$ such that $\bar{\alpha}\mathbf{x}^* \gg \mathbf{x} \geq \mathbf{0}$. This is possible because $\mathbf{x}^* \gg \mathbf{0}$.

Define the sets

$$A^+ = \left\{\alpha \in [0, \bar{\alpha}] : f(\alpha\mathbf{x}^*) \geq f(\mathbf{x})\right\}$$

and

$$A^- = \left\{\alpha \in [0, \bar{\alpha}] : f(\mathbf{x}) \geq f(\alpha\mathbf{x}^*)\right\}.$$

By continuity of f, both A^+ and A^- are closed subsets of $\mathbb{R}^m_+$. Further, they obey $A^+ \cup A^- \supset [0, \bar{\alpha}]$ because $f(\alpha\mathbf{x}^*)$ is defined for every $\alpha \geq 0$. Now $0 \in A^-$ and $\bar{\alpha} \in A^+$, so both are non-empty. Since intervals are connected, $A^+ \cap A^-$ must be non-empty, showing there is an α with $f(\alpha\mathbf{x}^*) = f(\mathbf{x})$.

We must still show there is only one such α. By construction, $f(\mathbf{x}) = f(\alpha\mathbf{x}^*)$ whenever $\alpha \in (A^+ \cap A^-)$. Now suppose there are α and α' with $f(\mathbf{x}) = f(\alpha\mathbf{x}^*) = f(\alpha'\mathbf{x}^*)$.

Label the two points so that $\alpha > \alpha'$. Then $\alpha\mathbf{x}^* \gg \alpha'\mathbf{x}^*$. Because f is increasing, $f(\alpha\mathbf{x}^*) > f(\alpha'\mathbf{x}^*)$. But this contradicts the fact that $f(\alpha\mathbf{x}^*) = f(\alpha'\mathbf{x}^*)$. This contradiction shows there is only one α with $f(\mathbf{x}) = f(\alpha\mathbf{x}^*)$. ∎

[5] See section 2.2.2 of my micro manuscript.

20.4.5 When Lemma 20.4.3 Fails

The fact that f is defined on every $\alpha\boldsymbol{x}^*$ played a key role in the proof. The lemma need not be true when f is defined on a disconnected set, such as $\mathbb{Q}^m_+$.

► **Example 20.4.4: Lemma 20.4.3 fails on $\mathbb{Q}^2_+$.** Take the function $f(x_1, x_2) = x_1 x_2$, which is defined on $\mathbb{Q}^2_+$ and set $\boldsymbol{x}^* = (1, 1)$. Then there is no rational number α with $\alpha\boldsymbol{x}^* \in \mathbb{Q}^2_+$ and

$$f(1, 2) = f(\alpha\boldsymbol{x}^*) = f(\alpha, \alpha) = \alpha^2.$$

The number that satisfies this equation is $\alpha = \sqrt{2}$, which is an irrational number. ◄

20.4.6 Homothetic Representation Theorem I

We are now ready to prove the Homothetic Representation Theorem.

Homothetic Representation Theorem. *Suppose* $f\colon \mathbb{R}^m_+ \to \mathbb{R}$ *is increasing, homothetic, and continuous on* $\mathbb{R}^m_+$. *Then* $f(\boldsymbol{x})$ *is a monotonic transformation of a homogeneous of degree one function* v.

Proof of Theorem. Apply Lemma 20.4.3 to $\boldsymbol{x}^* = \boldsymbol{e}$. Then for every $\boldsymbol{x} \in \mathbb{R}^m_+$, there is a $\alpha \geq 0$ with $f(\alpha\boldsymbol{e}) = f(\boldsymbol{x})$. In that case, define $v(\boldsymbol{x}) = \alpha$, so that $f\big(v(\boldsymbol{x})\boldsymbol{e}\big) = f(\boldsymbol{x})$.

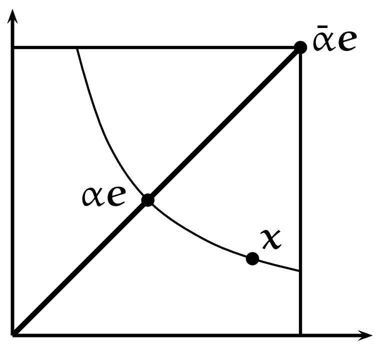

Figure 20.4.5: The position of the intersection of the indifference curve with the diagonal determines the unique α with $f(\boldsymbol{x}) = f(\alpha\boldsymbol{e})$.

Now $v(\boldsymbol{x}) \geq v(\boldsymbol{y})$ if and only if

$$f(\boldsymbol{x}) = f\big(v(\boldsymbol{x})\boldsymbol{e}\big) \geq f\big(v(\boldsymbol{y})\boldsymbol{e}\big) = f(\boldsymbol{y}).$$

It follows that v is ordinally equivalent to f.

Proof continues ...

20.4.7 Homothetic Representation Theorem II

Remainder of Proof. We know $f(\mathbf{x}) = f\big(v(\mathbf{x})\mathbf{e}\big)$. By homotheticity of f, $f(t\mathbf{x}) = f\big(tv(\mathbf{x})\mathbf{e}\big)$. But $f(t\mathbf{x}) = f\big(v(t\mathbf{x})\mathbf{e}\big)$, so

$$f\big(tv(\mathbf{x})\mathbf{e}\big) = f\big(v(t\mathbf{x})\mathbf{e}\big). \tag{20.4.2}$$

It follows that $tv(\mathbf{x}) = v(t\mathbf{x})$ because Lemma 20.4.3 showed that equation (20.4.2) has a unique solution. It follows that v is homogeneous of degree one.

Since f and v are equivalent, by Theorem 20.3.5, there is a monotonic transformation ϕ with $f(\mathbf{x}) = (\phi \circ v)(\mathbf{x})$. ∎

20.4.8 Homotheticity is not Homogenity

Although homothetic functions are related to homogeneous functions, there are differences. Nothing like Euler's Theorem need hold for functions that are merely homothetic. To see this, consider $f(\mathbf{x}) = \sum_\ell \alpha_\ell \ln x_\ell$. Then

$$\big[D_{\mathbf{x}} f(\mathbf{x})\big]\mathbf{x} = \sum_\ell \alpha_\ell,$$

which is not only different from the original function, but is even independent of $\mathbf{x}$.

December 2, 2022