21. Concave and Quasiconcave Functions

NB: The final will be given at 5pm on Tuesday, Dec. 6 in DM-164.

Convex, concave, and related functions arise naturally in economics. They include the indirect utility function, cost function, expenditure function, and profit function. Moreover, concavity is usually assumed of utility as it ensures a diminishing (or at least non-increasing) marginal rate of substitution. It often applies to production.

21.1 Convex and Concave Functions

Recall that in any vector space, the *line segment between $\mathbf{x}$ and $\mathbf{y}$* is given by $\ell(\mathbf{x}, \mathbf{y}) = \{(1-t)\mathbf{x} + t\mathbf{y} : 0 \le t \le 1\}$, and that a set S is *convex* if it contains $\ell(\mathbf{x}, \mathbf{y})$ whenever $\mathbf{x}, \mathbf{y} \in S$.

Convex and concave functions are only defined when the domain is convex.

Convex and Concave Functions. Let S be a convex set.

- A function $f\colon S \to \mathbb{R}$ is *convex* if for all $\mathbf{x}, \mathbf{y} \in S$ and $0 \le t \le 1$, $f\big(t\mathbf{x} + (1-t)\mathbf{y}\big) \le tf(\mathbf{x}) + (1-t)f(\mathbf{y})$.
- A function $f\colon S \to \mathbb{R}$ is *concave* if for all $\mathbf{x}, \mathbf{y} \in S$ and $0 \le t \le 1$, $f\big(t\mathbf{x} + (1-t)\mathbf{y}\big) \ge tf(\mathbf{x}) + (1-t)f(\mathbf{y})$.

When the inequalities are strict for $0 < t < 1$, we say the function is *strictly convex* or *strictly concave*.

One consequence is that for convex functions, every chord of the graph of the function lies on or above the graph. For concave functions, every chord lies on or below the graph.

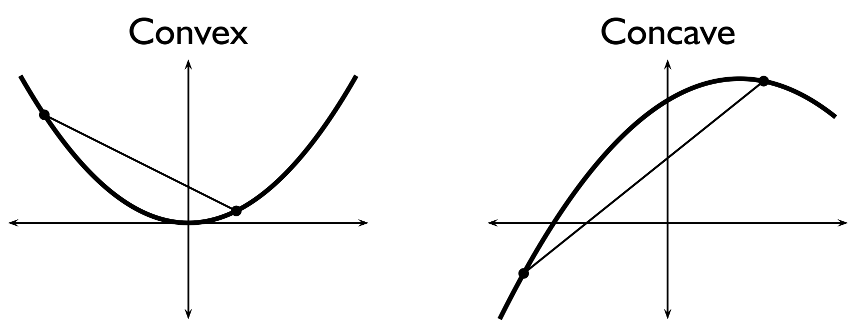

Figure 21.1.1: The left panel shows a convex function, where every chord connecting any two points of the graph lies above the graph.

The right panel illustrates a concave function, and every chord connecting any two points of the graph lies below the graph.

21.1.1 Can a Function be Both Convex and Concave?

Yes! Both linear and affine functions are both convex and concave.

► **Example 21.1.2: Convex Functions.** Any linear function $f(\mathbf{x}) = \mathbf{p}\cdot\mathbf{x}$ is both concave and convex, as is the generic affine function $f(\mathbf{x}) = a + \mathbf{p}\cdot\mathbf{x}$. The function $f(\mathbf{x}) = \sum_{\ell=1}^{m} x_\ell^2$ is convex while $f(\mathbf{x}) = \sum_{\ell=1}^{m} x_\ell^{1/2}$ is concave. The function $f(x) = e^x$ is convex, while $f(x) = \ln x$ is concave. ◄

It's also possible to have flat spots in the graph of a convex or concave function.

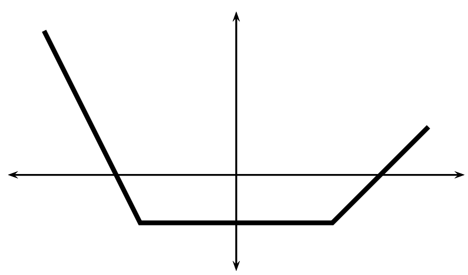

Figure 21.1.3: This function is convex, but not strictly convex. The flat portions of the graph rule out strict convexity.

21.1.2 Basic Properties of Concave and Convex Functions

Several easily established properties of convex and concave functions are collected together without proof in Theorem 21.1.4.

Theorem 21.1.4.

1. *A function* f *is (strictly) convex if and only if* $-f$ *is (strictly) concave.*
2. *A positive scalar multiple of a concave (convex) function is concave (convex).*
3. *The sum of two concave (convex) functions is concave (convex).*
4. *If* ϕ *is concave (convex) and weakly increasing on* $\mathbb{R}$ *and* f *is a concave (convex) function, then* $\phi \circ f$ *is concave (convex).*
5. *The pointwise limit of a sequence of concave (convex) functions is concave (convex).*
6. *The infimum (supremum) of a sequence of concave (convex) functions is concave (convex).*

Proof. You should be able to prove these yourself.

21.1.3 Upper and Lower Contour Sets

Let $f\colon S \to \mathbb{R}$ be a function where $S \subset \mathbb{R}^m$. The *upper contour set* is defined by $U(\mathbf{y}) = \{\mathbf{x} \in S : f(\mathbf{x}) \geq f(\mathbf{y})\}$. The *lower contour set* is defined by $L(\mathbf{y}) = \{\mathbf{x} \in S : f(\mathbf{x}) \leq f(\mathbf{y})\}$.

▶ **Example 21.1.5: A Convex Upper Contour Set.** Figure 21.5.2 illustrates an upper contour set for $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ given by $u(x, y) = xy$.

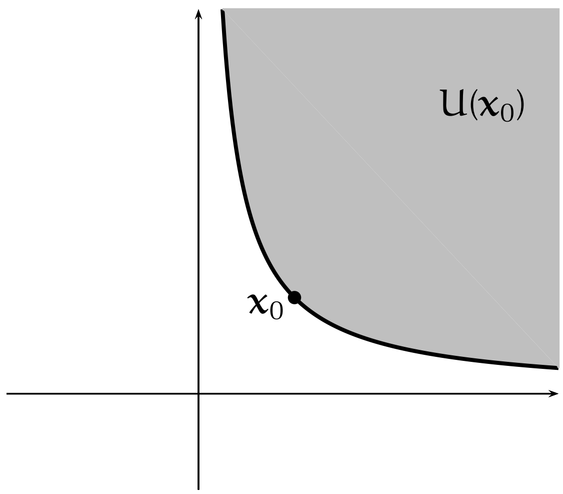

Figure 21.1.6: The shaded area is the upper contour set $U(\mathbf{x}_0)$ for $u(x, y) = xy$.

◀

21.1.4 Supporting Hyperplanes

An important concept in convex analysis is the supporting hyperplane.

Supporting Hyperplane. We say a vector $\mathbf{p}$ *supports a set* S at $\mathbf{x}_0 \in S$ if either $\mathbf{p}\cdot\mathbf{x} \geq \mathbf{p}\cdot\mathbf{x}_0$ for every $\mathbf{x} \in S$ or $\mathbf{p}\cdot\mathbf{x} \leq \mathbf{p}\cdot\mathbf{x}_0$ for every $\mathbf{x}$ in S.

Of course, the set $H = \{\mathbf{x} : \mathbf{p}\cdot\mathbf{x} = \mathbf{p}\cdot\mathbf{x}_0\}$ is a hyperplane. So supporting the set means that the set is on one side of the hyperplane. Moreover, they necessarily touch at $\mathbf{x}_0$.

That means we can restate the definition in terms of half-spaces. A vector $\mathbf{p}$ supports S at $\mathbf{x}_0$ if and only if S is contained in the one of the two half-spaces $H^+(\mathbf{p}, \mathbf{p}\cdot\mathbf{x}_0)$ and $H^-(\mathbf{p}, \mathbf{p}\cdot\mathbf{x}_0)$. Recall that

$$H^+(\mathbf{p}, \mathbf{p}\cdot\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{p}\cdot\mathbf{x} \geq \mathbf{p}\cdot\mathbf{x}_0\}.$$

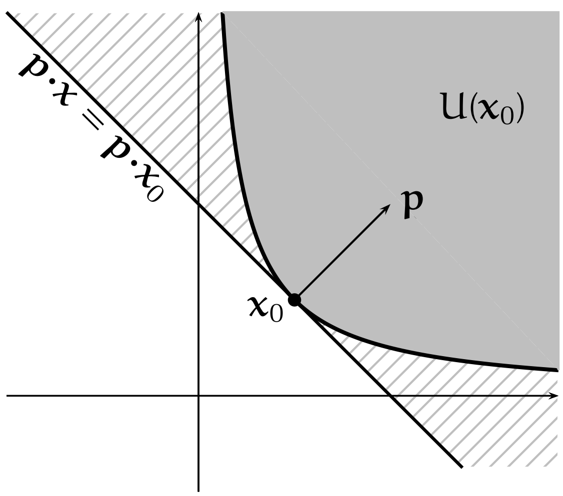

Figure 21.1.7: The shaded area is the upper contour set $U(\mathbf{x}_0)$ for $u(x, y) = xy$. The half-space $H^+(\mathbf{p}, \mathbf{x}_0)$ is the hatched area when $\mathbf{x}_0 = (1, 1)$ and $\mathbf{p} = (1, 1)$. The vector $\mathbf{p}$ supports $U(\mathbf{x}_0)$ at $\mathbf{x}_0$.

21.2 Support Property Theorem

The Support Property Theorem shows that a differentiable function f is concave if and only if $Df(\mathbf{x}_0)$ supports the upper contour set at every point where $f(\mathbf{x})$ is defined, for every $\mathbf{x}_0 \in \operatorname{dom} f$. It also shows that f is convex if and only if $Df(\mathbf{x}_0)$ supports the lower contour set at $\mathbf{x}_0$ for every $\mathbf{x}_0 \in \operatorname{dom} f$.

Support Property Theorem. *Suppose $f\colon U \to \mathbb{R}$ is $\mathcal{C}^1$ where U is an open convex set $U \subset \mathbb{R}^m$. The function f is concave if and only if*

$$f(\mathbf{y}) \le f(\mathbf{x}) + \big[Df(\mathbf{x})\big](\mathbf{y}-\mathbf{x}) \tag{21.2.1}$$

for all $\mathbf{x}, \mathbf{y} \in U$. The function f is convex if and only if

$$f(\mathbf{y}) \ge f(\mathbf{x}) + \big[Df(\mathbf{x})\big](\mathbf{y}-\mathbf{x}) \tag{21.2.2}$$

for all $\mathbf{x}, \mathbf{y} \in U$.

21.2.1 Proof of Support Property I

Proof (Only If). We must show that the inequality 21.2.1 holds when f is concave. Suppose f is concave and take ε with $0 < \varepsilon < 1$.

$$f\big(\mathbf{x} + \varepsilon(\mathbf{y} - \mathbf{x})\big) = f\big((1-\varepsilon)\mathbf{x} + \varepsilon\mathbf{y}\big) \geq \varepsilon f(\mathbf{y}) + (1-\varepsilon)f(\mathbf{x}).$$

We can rearrange to obtain

$$f\big(\mathbf{x} + \varepsilon(\mathbf{y} - \mathbf{x})\big) - f(\mathbf{x}) \geq \varepsilon\big[f(\mathbf{y}) - f(\mathbf{x})\big].$$

Dividing by $\varepsilon > 0$ and letting $\varepsilon \to 0$ yields

$$\big[Df(\mathbf{x})\big](\mathbf{y} - \mathbf{x}) \geq f(\mathbf{y}) - f(\mathbf{x}).$$

When f is convex, the only change to this part of the proof is that all three inequalities must be reversed.

Proof continues . . .

21.2.2 Proof of Support Property II

Proof (If). Now suppose the inequality 21.2.1 holds. We must show that f is concave.

$$f(\mathbf{y}) \leq f(\mathbf{x}) + \left[Df(\mathbf{x})\right](\mathbf{y} - \mathbf{x}) \tag{21.2.1}$$

is satisfied for all $\mathbf{x}, \mathbf{y} \in U$. Replace $\mathbf{x}$ by $\mathbf{x}' = \mathbf{x} + (1-\alpha)(\mathbf{y} - \mathbf{x}) = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}$ where $0 \leq \alpha \leq 1$. It follows that $\mathbf{y} - \mathbf{x} = \alpha(\mathbf{y} - \mathbf{x})$. By convexity of U, $\mathbf{x}' \in U$. Then

$$\begin{aligned} f(\mathbf{y}) &\leq f(\mathbf{x}') + \alpha\left[Df(\mathbf{x}')\right](\mathbf{y} - \mathbf{x}) \\ &\leq f(\mathbf{x}') - \alpha\left[Df(\mathbf{x}')\right](\mathbf{x} - \mathbf{y}). \end{aligned} \tag{21.2.3}$$

Rewrite the support equation (21.2.1) by replacing $\mathbf{x}$ with $\mathbf{x}'$ and $\mathbf{y}$ with $\mathbf{x}$. Now $\mathbf{x} - \mathbf{x}' = (1-\alpha)(\mathbf{x} - \mathbf{y})$. This yields

$$f(\mathbf{x}) \leq f(\mathbf{x}') + (1-\alpha)\left[Df(\mathbf{x}')\right](\mathbf{x} - \mathbf{y}). \tag{21.2.4}$$

Then multiply equation (21.2.3) by $(1-\alpha)$, and multiply equation (21.2.4) by α, obtaining

$$\begin{aligned} (1-\alpha)f(\mathbf{y}) &\leq (1-\alpha)f(\mathbf{x}') - \alpha(1-\alpha)\left[Df(\mathbf{x}')\right](\mathbf{x} - \mathbf{y}) \quad \text{and} \\ \alpha f(\mathbf{x}) &\leq \qquad \alpha f(\mathbf{x}') + \alpha(1-\alpha)\left[Df(\mathbf{x}')\right](\mathbf{x} - \mathbf{y}). \end{aligned}$$

Add them together. The $Df(\mathbf{x}')$ terms cancel, leaving

$$\alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}) \leq f(\mathbf{x}') = f\big(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\big)$$

establishing concavity. The proof for the convex case is the same, but with every inequality reversed. ∎

21.2.3 Supporting Upper and Lower Contour Sets

One consequence of the Support Property Theorem is that if f is concave, the derivative $Df(\mathbf{x}_0)$ supports the upper contour set $U(\mathbf{x}_0)$ at $\mathbf{x}_0$. To see this, suppose $f(\mathbf{y}) \geq f(\mathbf{x}_0)$. Then equation (21.2.1) implies

$$0 \leq f(\mathbf{y}) - f(\mathbf{x}_0) \leq \big[Df(\mathbf{x}_0)\big](\mathbf{y} - \mathbf{x}_0),$$

so $Df(\mathbf{x}_0)\mathbf{y} \geq Df(\mathbf{x}_0)\mathbf{x}_0$. This implies $U(\mathbf{x}_0) \subset H^+\big(Df(\mathbf{x}_0), \mathbf{x}_0\big)$ as in Figure 21.1.7.

Similarly, if f is convex, the derivative $Df(\mathbf{x}_0)$ supports the lower contour set $L(\mathbf{x}_0)$ at $\mathbf{x}_0$.

21.2.4 The Support Property in $\mathbb{R}$

By the Support Property Theorem, a differentiable function is concave if and only if equation (21.2.1) holds. This important inequality can be taken as the definition of concavity for differentiable functions.

$$f(\mathbf{y}) \le f(\mathbf{x}) + \big[Df(\mathbf{x})\big](\mathbf{y} - \mathbf{x}) \tag{21.2.1}$$

When f is a function on a subset of the real line, we can use the right-hand side of equation (21.2.1) to define a line, $y = f(x_0) + f'(x_0)(x - x_0)$. This line is tangent to the graph of f at the point $\big(x_0, f(x_0)\big)$, and the graph of the function is in the lower half-space that the tangent line defines. The tangent line supports the graph of f in the sense that the graph lies within one of the half-spaces defined by the tangent.

The concave **conjugate function** $f^*(p)$ is the negative of the vertical intercept of the tangent line. Although we won't explore it further at this time, it plays an important role in economic duality.

The tangent line supports both the graph and subgraph of f at $\big(x_0, f(x_0)\big)$. The subgraph is

$$\text{sub}\, f = \big\{(\mathbf{x}, y) : f(\mathbf{x}) \le y\big\}.$$

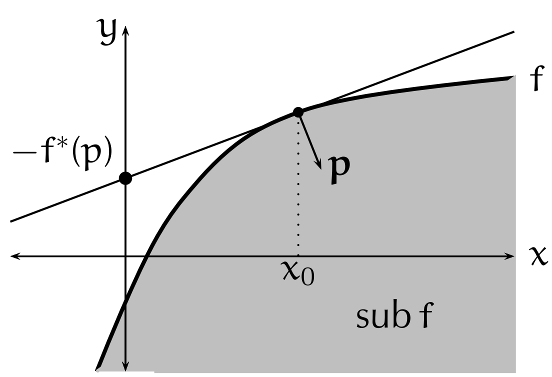

Figure 21.2.1: The tangent line at x_0 has the equation $y = f(x_0) + f'(x_0)(x - x_0)$. Because f is concave, the tangent line supports the subgraph of f. The graph is never above the tangent line and touches it at $\big(x_0, f(x_0)\big)$. Let $p = f'(x_0)$. The vector $\mathbf{p} = (p, -1)$ is perpendicular to the tangent line. Its vertical intercept is the negative of the concave conjugate function $f^*(p) = px_0 - f(x_0)$.

21.2.5 Support Property in $\mathbb{R}^m$

In Figure 21.2.1, $p = f'(x_0)$ is the slope of the tangent line. We now rewrite the equation of the tangent in a way that that shows it is a hyperplane in $\mathbb{R}^2$. Thus

$$y - f(x_0) = p(x - x_0)$$

is rewritten as

$$(p, -1)\begin{pmatrix} x \\ y \end{pmatrix} = px_0 - f(x_0). \tag{21.2.5}$$

The vector $\mathbf{p} = (p, -1)$ is perpendicular to the tangent line, which is parallel to $(1, p)$, meaning that p is the slope of the tangent.

The right-hand side of equation (21.2.5) is not zero unless tangent goes through the origin. It tells us how much the tangent line is offset from the origin. That value is called the *concave conjugate function* and is denoted $f^*(p) = px_0 - f(x_0)$ when $p = f'(x_0)$. In fact, $f^*(p)$ is the negative of the vertical intercept of the tangent line. The equation of the tangent line then becomes

$$(p, -1)\begin{pmatrix} x \\ y \end{pmatrix} = f^*(p)$$

and the support inequality can now be written

$$(p, -1)\begin{pmatrix} x \\ f(x) \end{pmatrix} \geq f^*(p). \tag{21.2.6}$$

More generally, if f is a function of m variables, we can consider its graph in $\mathbb{R}^{m+1}$, and the picture is much the same.

$$\big(Df(\mathbf{x}_0), -1\big)\begin{pmatrix} \mathbf{x} \\ f(\mathbf{x}) \end{pmatrix} \leq \big(Df(\mathbf{x}_0), -1\big)\begin{pmatrix} \mathbf{x}_0 \\ f(\mathbf{x}_0) \end{pmatrix}$$

where $\mathbf{p} = Df(\mathbf{x}_0)$.

21.3 Support Property and the Hessian

For $\mathcal{C}^2$ functions, the support property allows us to relate convexity or concavity and the properties of the Hessian and its principal minors.

21.3.1 Hessian Convexity Tests: Necessity

The support property can be used to show that the Hessian $D^2f(\mathbf{x})$ is negative semidefinite when f is concave and positive semidefinite when f is convex.

Theorem 21.3.1. *Suppose* $f\colon U \to \mathbb{R}$ *is* $\mathcal{C}^2$ *on an open convex set* $U \subset \mathbb{R}^m$. *If* f *is concave, then the Hessian* $D^2f(\mathbf{x})$ *is negative semidefinite for all* $\mathbf{x} \in U$. *If* f *is convex, then the Hessian* $D^2f(\mathbf{x})$ *is positive semidefinite for all* $\mathbf{x} \in U$.

Proof. We will prove the concave case, the convex case is similar, with inequalities reversed. By the Support Property Theorem (equation 21.2.1)

$$f(\mathbf{y}) \le f(\mathbf{x}) + Df(\mathbf{x})(\mathbf{y}-\mathbf{x}) = f(\mathbf{x}) - Df(\mathbf{x})(\mathbf{x}-\mathbf{y})$$

reversing $\mathbf{x}$ and $\mathbf{y}$ in equation (21.2.1) yields

$$f(\mathbf{x}) \le f(\mathbf{y}) + Df(\mathbf{y})(\mathbf{x}-\mathbf{y}).$$

Adding the equations together and simplifying, we obtain

$$0 \le \big(Df(\mathbf{y}) - Df(\mathbf{x})\big)(\mathbf{x}-\mathbf{y}).$$

Now set $\mathbf{y} = \mathbf{x} + \mathbf{h}$. We then have

$$\big(Df(\mathbf{x}+\mathbf{h}) - Df(\mathbf{x})\big)\mathbf{h} \le 0.$$

Divide by $\|\mathbf{h}\|$ and let $\|\mathbf{h}\| \to 0$, obtaining $\mathbf{h}^T\big[D^2f(\mathbf{x})\big]\mathbf{h} \le 0$. In other words, $D^2f(\mathbf{x})$ is negative semidefinite for all $\mathbf{x} \in U$. ∎

21.3.2 Hessian Convexity Tests: Necessity and Sufficiency

When f is $\mathcal{C}^2$, we can use the Hessian matrix and Taylor's formula to determine whether f is concave or convex.

Theorem 21.3.2. *Suppose* $f\colon U \to \mathbb{R}$ *is* $\mathcal{C}^2$ *on an open convex set* $U \subset \mathbb{R}^m$.

1. *The function* f *is concave if and only if* $D^2 f(\mathbf{x})$ *is negative semidefinite for all* $\mathbf{x} \in U$.
2. *If* $D^2 f(\mathbf{x})$ *is negative definite for all* $\mathbf{x} \in U$, f *is strictly concave.*
3. *The function* f *is convex if and only if* $D^2 f(\mathbf{x})$ *is positive semidefinite for all* $\mathbf{x} \in U$.
4. *If* $D^2 f(\mathbf{x})$ *is positive definite for all* $\mathbf{x} \in U$, f *is strictly convex.*

Proof. Taylor's formula tells us that

$$f(\mathbf{y}) = f(\mathbf{x}) + \big[Df(\mathbf{x})\big](\mathbf{y} - \mathbf{x}) + (\mathbf{y} - \mathbf{x})^{\mathsf{T}}\big[D^2 f(\mathbf{z})\big](\mathbf{y} - \mathbf{x}). \qquad (21.3.7)$$

for some $\mathbf{z}$ on the line segment between $\mathbf{x}$ and $\mathbf{y}$.

(1) If $D^2 f$ is negative semidefinite on U, this implies the support property, so f is concave by the Support Property Theorem. Conversely, if f is concave, Proposition 21.3.1 shows that $D^2 f(\mathbf{x})$ is negative semidefinite on U.

(2) If f is negative definite on U, equation (21.3.7) yields $f(\mathbf{y}) < f(\mathbf{x}) + \big[Df(\mathbf{x})\big](\mathbf{y} - \mathbf{x})$ for all $\mathbf{y} \neq \mathbf{x}$. Repeating the calculations in the Support Property Theorem for $\mathbf{y} \neq \mathbf{x}$ and $0 < \alpha < 1$, shows that f is strictly concave.

Parts (3) and (4) are similar. ∎

21.3.3 Definiteness and Determinants Reviewed

Recall the following definitions

Definite and Semidefinite Matrices. Recall that an $m \times m$ symmetric matrix $\mathbf{A}$ is

(a) *Positive semidefinite* if $\mathbf{x}^T\mathbf{A}\mathbf{x} \le 0$ for all $\mathbf{x} \in \mathbb{R}^m$,

(b) *Positive definite* if $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$,

(c) *Negative semidefinite* if $\mathbf{x}^T\mathbf{A}\mathbf{x} \le 0$ for all $\mathbf{x} \in \mathbb{R}^m$,

(d) *Negative definite* if $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$, and

(e) *Indefinite* if there are $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ and $\mathbf{y}^T\mathbf{A}\mathbf{y} < 0$.

In Theorem 21.3.1 we used the support property to show that $D^2f(\mathbf{x})$ is negative semidefinite when f is concave and positive semidefinite when f is convex.

If $\mathbf{A}$ is a matrix, we will use $\tilde{\mathbf{A}}_k$ to denote a generic k^{th}-order principal minor of $\mathbf{A}$. One example of a non-leading principal minor for $m = 6$ and $k = 3$ is

$$\begin{vmatrix} a_{11} & a_{13} & a_{16} \\ a_{31} & a_{33} & a_{36} \\ a_{61} & a_{63} & a_{66} \end{vmatrix}$$

where the 2^{nd}, 4^{th}, and 5^{th} rows and columns have been deleted. Keep in mind that there are usually many k^{th}-order principal minors.

21.3.4 Convexity and Concavity: Determinant Tests

We can rewrite our results relating convexity and definiteness by applying the usual determinant tests to the Hessian. That yields the following theorem.

Theorem 21.3.3. *Suppose* $f\colon U \to \mathbb{R}$ *is* $\mathcal{C}^2$ *on an open convex set* $U \subset \mathbb{R}^m$. *Let* $\mathbf{H}(\mathbf{x}) = D^2f(\mathbf{x})$ *denote the Hessian of* f.

1. *The function* f *is convex if and only if every* k^{th}*-order principal minor obeys* $\tilde{\mathbf{H}}_k(\mathbf{x}) \geq 0$ *for all* $\mathbf{x} \in U$.
2. *The function* f *is concave if and only if every* k^{th}*-order principal minor obeys* $(-1)^k\tilde{\mathbf{H}}_k(\mathbf{x}) \geq 0$ *for all* $\mathbf{x} \in U$.
3. *Suppose the leading principal minors obey* $H_k(\mathbf{x}) > 0$ *for* $k = 1, \ldots, m$ *and all* $\mathbf{x} \in U$. *Then* f *is strictly convex.*
4. *Suppose the leading principal minors obey* $(-1)^k H_k(\mathbf{x}) > 0$ *for* $k = 1, \ldots, m$ *and all* $\mathbf{x} \in U$. *Then* f *is strictly concave.*

21.3.5 Using the Determinant Test

Here's how it works for Cobb-Douglas functions on $\mathbb{R}^2_{++}$.

▶ **Example 21.3.4: Cobb-Douglas Utility.** Consider the Cobb-Douglas utility function on $\mathbb{R}^2_{++}$ defined by $u(\mathbf{x}) = x_1^{\alpha} x_2^{1-\alpha}$ with $0 < \alpha < 1$. The Hessian is

$$\begin{aligned} \mathbf{H}(\mathbf{x}) &= \begin{bmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} \\ \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & (1-\alpha)(-\alpha)x_1^{\alpha}x_2^{-\alpha-1} \end{bmatrix} \\ &= \alpha(1-\alpha)\begin{bmatrix} -x_1^{\alpha-2}x_2^{1-\alpha} & x_1^{\alpha-1}x_2^{-\alpha} \\ x_1^{\alpha-1}x_2^{-\alpha} & -x_1^{\alpha}x_2^{-\alpha-1} \end{bmatrix}. \end{aligned}$$

Clearly, $\det \mathbf{H} = 0$. The two first-order minors are $-\alpha(1-\alpha)x_1^{\alpha-2}x_2^{1-\alpha} < 0$ and $-\alpha(1-\alpha)x_1^{\alpha}x_2^{-\alpha-1} < 0$, which shows that the Hessian is negative semidefinite on $\mathbb{R}^2_{++}$. This implies that u is concave on $\mathbb{R}^2_{++}$. Concavity on $\mathbb{R}^2_{+}$ then follows from continuity.

The function u is not strictly concave because $\alpha u(\mathbf{0}) + (1-\alpha)u(\boldsymbol{e}) = 1-\alpha = u\big((1-\alpha)\boldsymbol{e}\big)$. ◀

Here's another example.

▶ **Example 21.3.5: CES Utility.** Another example is the constant elasticity of substitution utility function $u(\mathbf{x}) = [x_1^{-\rho} + x_2^{-\rho}]^{-1/\rho}$ where $\rho > -1$, $\rho \neq 0$. The Hessian is

$$\mathbf{H}(\mathbf{x}) = (1+\rho)\frac{[x_1^{-\rho} + x_2^{-\rho}]^{-(\frac{1+2\rho}{\rho})}}{x_1^{2+\rho}x_2^{2+\rho}}\begin{pmatrix} -x_2^2 & x_1x_2 \\ x_1x_2 & -x_1^2 \end{pmatrix}.$$

The two first-order minors are both negative, while the second-order minor is 0. Thus u is concave on $\mathbb{R}^2_{++}$. ◀

21.4 Non-Differentiable Functions and the Supergradient

We can often support concave functions when they are not differentiable. However, there may be more than one vector $\mathbf{p}$ that supports them in such a case. Obviously, the derivative cannot define the supporting hyperplane. However, derivatives can be generalized using supergradients and subgradients.

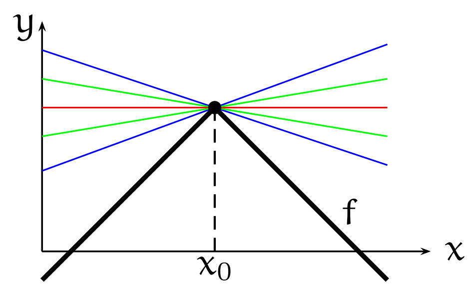

Figure 21.4.1: As you can see, there are many lines that support the function f at $(x_0, f(x_0))$. The function has slope $+1$ to the left of x_0 and -1 to the right. Any line with a slope between -1 and $+1$ that is tangent to the graph of f at $(x_0, f(x_0))$ will satisfy the support property.

As we will see, the function in Figure 21.4.1 has a supergradient at $\mathbf{x}_0$, even though it is not differentiable there.

21.4.1 Supergradients and Subgradients

We can define generalized derivatives for situations like Figure 21.4.1.

Supergradient. Let f be a concave function on a convex set $U \subset \mathbb{R}^m$, and $\mathbf{x}$ a point in U. If $\mathbf{p} \in \mathbb{R}^m$ satisfies

$$f(\mathbf{y}) \leq f(\mathbf{x}) + \mathbf{p}\cdot(\mathbf{y} - \mathbf{x}), \tag{21.4.8}$$

for all $\mathbf{y} \in U$, we call $\mathbf{p}$ a *supergradient* of f at $\mathbf{x}$.

The supergradient is a type of generalized derivative. Figure 21.4.1 illustrates this. The function there is not differentiable when $x = 0$, but it does have supergradients there. Any $p \in [-1, +1]$ is a supergradient at $x = 0$.

There's also a similar generalization for convex functions, called a subgradient.

Subgradient. Let f be a convex function on a convex set $U \subset \mathbb{R}^m$, and $\mathbf{x}$ a point in U. If $\mathbf{p} \in \mathbb{R}^m$ satisfies

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{p}\cdot(\mathbf{y} - \mathbf{x}), \tag{21.4.9}$$

for all $\mathbf{y} \in U$, we call $\mathbf{p}$ a *subgradient* of f at $\mathbf{x}$.

The Support Function Theorem tells us that when f is differentiable and concave (convex) $Df(\mathbf{x}_0)$ is a supergradient (subgradient) of f at $\mathbf{x}_0$. In fact, it will turn out to be the only one up to scalar multiplication (see Theorem 21.6.2).

21.4.2 Supergradients and Optimization

When f is concave (convex) the first order conditions are not only necessary for an optimum, they are sufficient too.

Theorem 21.4.2. *Let $U \subset \mathbb{R}^m$ be convex.*

(a) *Suppose $\mathbf{0}$ is a supergradient of f at $\mathbf{x}^*$. Then $\mathbf{x}^*$ is a global maximizer of f over U. In particular, this applies if f is differentiable and $Df(\mathbf{x}^*) = \mathbf{0}$.*

(b) *Suppose $\mathbf{0}$ is a subgradient of f at $\mathbf{x}^*$. Then $\mathbf{x}^*$ is a global minimizer of f over U. In particular, this applies if f is differentiable and $Df(\mathbf{x}^*) = \mathbf{0}$.*

Proof. Setting $\mathbf{p} = \mathbf{0}$ in the supergradient inequality (21.4.8), yields $f(\mathbf{y}) \leq f(\mathbf{x}^*)$ for all $\mathbf{y} \in U$. ∎

This even works on some concave functions that aren't differentiable, as in Figure 21.4.1 where a horizontal line ($p = 0$) supports the function at x_0. That means x_0 is a maximum, as is also clear from the graph.

This result is extremely useful in economics, as we are faced with a convex feasible set U and a concave utility function we wish to maximize or convex function such as $\boldsymbol{w}\cdot\boldsymbol{z}$ needing to be minimized. In either case, the first order necessary conditions become sufficient for optimization.

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21.4.3 Convexity and Differentiable Optimization

We can go a bit further than this for differentiable functions.

Theorem 21.4.3. *Let $U \subset \mathbb{R}^m$ be convex.*

(a) *Suppose f is a concave $\mathcal{C}^1$ function on U and $\mathbf{x}^*$ obeys $Df(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) \leq 0$ for all $\mathbf{y} \in U$. Then $\mathbf{x}^*$ is a global maximizer of f on U.*

(b) *Suppose f is a convex $\mathcal{C}^1$ function on U and $\mathbf{x}^*$ obeys $Df(\mathbf{x}^*)(\mathbf{y}-\mathbf{x}^*) \geq 0$ for all $\mathbf{y} \in U$. Then $\mathbf{x}^*$ is a global minimizer of f on U.*

Proof. For the concave case, if $Df(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) \leq 0$, then $f(\mathbf{y}) \leq f(\mathbf{x}^*) + Df(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) \leq f(\mathbf{x}^*)$ by the Support Property Theorem. The convex case is similar. ∎

► **Example 21.4.4: Decreasing Functions.** Suppose f is a decreasing $\mathcal{C}^1$ function of one variable on an interval $[a, b]$. Then $f'(a)(x - a) \leq 0$ for all $x \in [a, b]$ because $x - a \geq 0$ and $f'(a) \leq 0$. It follows that a is a maximizer of f on $[a, b]$ ◄

It works in $\mathbb{R}^m$ too. Here's an example for $\mathbb{R}^2$.

► **Example 21.4.5: Global Minimum via Support.** Let $f(x, y) = -x^2 - y^2$ on the set $[0, 4] \times [0, 3] \subset \mathbb{R}^2$. Then $Df = (-2x, -2y)$. We will show that $\mathbf{x}^* = (4, 3)$ is a minimizer. Now $Df(\mathbf{x}^*) = (-8, -6)$ and $Df(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) = (-8, -6)\cdot(x - 4, y - 3)$. Here both $x \leq 4$ and $y \leq 3$, so $Df(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \geq 0$, showing that $\mathbf{x}^*$ is a global minimum. ◄

21.5 Quasiconcave and Quasiconvex Functions

Quasiconcavity and quasiconvexity are ordinal concepts closely related to concavity and convexity. They are defined in terms of the upper and lower contour sets, respectively. We will see that upper and lower contour sets are unaffected by monotonic transformations, and hence ordinal.

21.5.1 Convexity of Upper and Lower Contour Sets I

Upper contour sets are convex for concave functions while lower contour sets are convex for convex functions (see next page). It is often the case that the lower contour set of a concave function is not convex. This happens in the figure below.

▶ Example 21.5.1: Upper and Lower Contour Sets. The left side Figure 21.5.2 illustrates an upper contour set for $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ given by $u(x, y) = x^{1/3}y^{2/3}$. The right side shows the lower contour set for the same function. It is not convex.

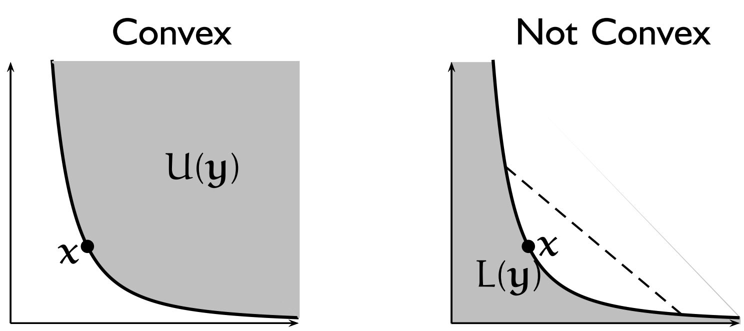

Figure 21.5.2: The shaded area in the left panel is the upper contour set $U(\mathbf{y})$ for $u(x, y) = x^{1/3}y^{2/3}$ and $\mathbf{y} = (1, 1)$.

The shaded area in right panel is the lower contour set for the same function. The dashed line segment would be in $L(\mathbf{y})$ if $L(\mathbf{y})$ were convex. The line segment goes outside $L(\mathbf{y})$, so the lower contour set is not convex.

◀

21.5.2 Convexity of Upper and Lower Contour Sets II

In Figure 21.5.2, the upper contour set was convex. This was no accident. Rather, it is a consequence of using a concave function to define the upper contour set.

Theorem 21.5.3. *Let* $U \subset \mathbb{R}^m$ *be a convex set.*

1. *If* $f\colon U \to \mathbb{R}$ *is a concave function, then the upper contour set* $U(y)$ *is convex.*
2. *If* $f\colon U \to \mathbb{R}$ *is a convex function, then the lower contour set* $L(y)$ *is convex.*

Proof. Let $x, x' \in U(y)$. Then $f(x), f(x') \geq f(y)$. By concavity of f

$$\begin{aligned} f\big((1-t)x + tx'\big) &\geq (1-t)f(x) + tf(x') \\ &\geq (1-t)f(y) + tf(y) \\ &= f(y), \end{aligned}$$

showing that the convex combination $(1-t)x + tx' \in U(y)$ whenever $x, x' \in U(y)$.

The proof of part (2) is similar. ∎

21.5.3 Contour Sets and Quasiconcavity/Quasiconvexity

We start by noting that contour sets are ordinal, unchanged by montotonic transformations of the functions.

Contour Sets are Ordinal. The sets $U(\mathbf{x})$ and $L(\mathbf{x})$ remain unchanged when f is replaced at an ordinal equivalent—when a monotonic transformation is applied to f. This is because $f(\mathbf{x}) \geq f(\mathbf{y})$ if and only if $(\phi \circ f)(\mathbf{x}) \geq (\phi \circ f)(\mathbf{y})$ when ϕ is a monotonic transformation.

We now use the upper and lower contour sets to define quasiconcavity and quasiconvexity.

Quasiconcavity and Quasiconvexity. Let $f\colon S \to \mathbb{R}$.

1. The function f is *quasiconcave* on S if and only if the upper contour set $U(\mathbf{y}) = \{\mathbf{x} \in S : f(\mathbf{x}) \geq f(\mathbf{y})\}$ is a convex set for every $\mathbf{y} \in S$.
2. The function f is *quasiconvex* on S if and only if the lower contour set $L(\mathbf{x}) = \{\mathbf{x} \in S : f(\mathbf{x}) \leq f(\mathbf{y})\}$ is a convex set for every $\mathbf{y} \in S$.

From the definition, it is easy to see that f is quasiconvex if and only if $-f$ is quasiconcave. Also, any increasing transformation of a concave (or quasiconcave) function is quasiconcave and any increasing transformation of a convex (or quasiconvex) function is quasiconvex.

21.5.4 Another Take on Quasiconcavity/Quasiconvexity

There are other ways to characterize quasiconcavity and quasiconvexity. One is the following.

Theorem 21.5.4. *Let* $f\colon S \to \mathbb{R}$ *where* S *is a convex set.*

1. *The function* f *is quasiconcave if and only if for all* t *obeying* $0 \le t \le 1$, $f(t\mathbf{x} + (1-t)\mathbf{y}) \ge \min\{f(\mathbf{x}), f(\mathbf{y})\}$.
2. *The function* f *is quasiconvex if and only if for all* t *obeying* $0 \le t \le 1$, $f(t\mathbf{x} + (1-t)\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\}$.

Of course, if $f\colon S \to \mathbb{R}$ is quasiconcave or quasiconvex on S, S must be a convex set because if $\mathbf{x}', \mathbf{x}'' \in S$, convex combinations of $\mathbf{x}'$ and $\mathbf{x}''$ are in either $\{\mathbf{x} \in S : f(\mathbf{x}) \ge \min[f(\mathbf{x}'), f(\mathbf{x}'')]\}$ (f quasiconcave) or in $\{\mathbf{x} \in S : f(\mathbf{x}) \le \max[f(\mathbf{x}'), f(\mathbf{x}'')]\}$ (f quasiconvex).

21.5.5 Monotone Functions on the Real Line

In Theorem 21.5.3 we saw that convex functions are quasiconvex and concave functions are quasiconcave. The relationship is only one-way.

Consider monotone functions on the real line. Such functions are either weakly increasing or weakly decreasing.[1]

Monotone functions on the real line are simultaneously quasiconvex and quasiconcave! This contrasts strongly with the fact that the only functions that are both convex and concave on $\mathbb{R}$ are the affine functions.

Theorem 21.5.5. *Let* I *be an interval in* $\mathbb{R}$ *and* f *a monotone function,* $f\colon I \to \mathbb{R}$. *Then* f *is both quasiconcave and quasiconvex.*

Proof. Suppose f is monotone increasing. Then

$$U(x) = \{y \in I : f(y) \geq f(x)\} = \{y \in I : y \geq x\}$$

and

$$L(x) = \{y \in I : f(y) \leq f(x)\} = \{y \in I : y \leq x\}$$

Both $U(x)$ and $L(x)$ are intervals in I, hence convex. This shows f is both quasiconcave and quasiconvex.

The same argument with the inequalities reversed applies when f is monotone decreasing. ∎

[1] Yes, monotone is used in a variety of ways.

21.5.6 Quasiconvexity on the Real Line I

For f defined on a convex subset of $\mathbb{R}$, we can completely characterize the quasiconcave and quasiconvex functions. Quasiconcave functions are either monotone or single peaked. Quasiconvex functions are either monotone or single troughed.

Theorem 21.5.6. *Let I be an interval in $\mathbb{R}$. Suppose $f\colon I \to \mathbb{R}$.*

1. *The function f is quasiconcave if and only f is either monotone or there exists a number x^* such that f is weakly increasing when $x \le x^*$ and weakly decreasing when $x \ge x^*$.*
2. *The function f is quasiconvex if and only if f is either monotone or there exists a number x^* such that f is weakly decreasing when $x \le x^*$ and weakly increasing when $x \ge x^*$.*

Proof. It is enough to prove the quasiconcave case as $-f$ is quasiconcave if f is quasiconvex.

If case. By Theorem 21.5.5, monotone functions are quasiconcave. We need only consider the case where x^* exists. Let $y \in I$ be given and let x_1 and x_2 be arbitrary points in the upper contour set $U(y)$ with $x_1 < x_2$. We must show that the interval $[x_1, x_2] \subset U(y)$.

If x^* is not in $[x_1, x_2]$, then f is either always weakly decreasing or always weakly increasing on $[x_1, x_2]$. Either way, for any $z \in [x_1, x_2]$, $f(z) \ge \min\{f(x_1), f(x_2)\} \ge f(y)$, so $z \in U(y)$. This shows that $[x_1, x_2] \subset U(y)$.

If $x^* \in [x_1, x_2]$, f is weakly increasing on $[x_1, x^*]$ and weakly decreasing on $[x^*, x_2]$. Again, for any $z \in [x_1, x_2]$, $f(z) \ge \min\{f(x_1), f(x_2)\} \ge f(y)$, so $z \in U(y)$ and hence $[x_1, x_2] \subset U(y)$. In other words, f is quasiconcave.

Proof continues . . .

21.5.7 Quasiconvexity on the Real Line II

Only if case. We prove this part by contradiction. Suppose that f is quasiconcave on I, but not monotone. Suppose further that x^* does not exist.

Then we can find x_1, x_2, and x_3 in I such that $x_1 < x_2 < x_3$ and $f(x_2) < \min\{f(x_1), f(x_3)\}$. If $f(x_1) \le f(x_3)$, set $y = x_1$, otherwise take $y = x_3$. Then the upper contour set $U(y)$ includes x_1 and x_3, but not x_2. It is not convex, contradicting the fact that f is quasiconcave. ∎

To sum up, quasiconcave functions defined on a real interval are either monotonic or single-peaked, while quasiconvex functions are either monotonic or single-troughed. Either the peak or trough may be an interval.

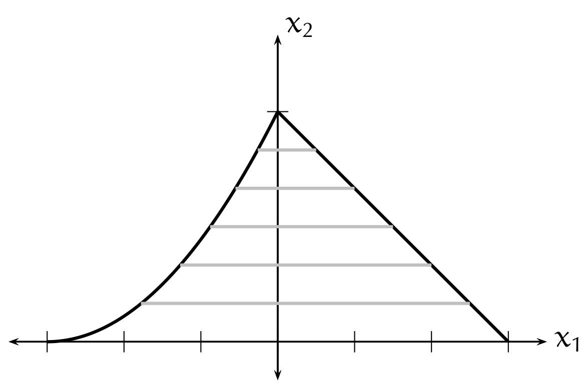

Figure 21.5.7: A quasiconcave function from $[-3, 3]$ to $\mathbb{R}$. Several upper contour sets are illustrated by the gray horizontal lines. If you project them into the x-axis, you get the actual upper contour sets, which are defined by $\{x : f(x) \ge y\}$ for $y = 0.5, 1, 1.5, 2$, and 2.5. They are all convex, as expected for a quasiconcave function.

With a little thought, you should be able to see that the lower contour sets, which have the form $\{x : f(x) \le y\}$ consist of two disjoint closed intervals for $0 < y < 3$. What happens at $y = 3$? $y = 0$?

21.5.8 Quasiconvexity on the Real Line III

Here's an example of a quasiconvex function on the interval $[-3, +3]$.

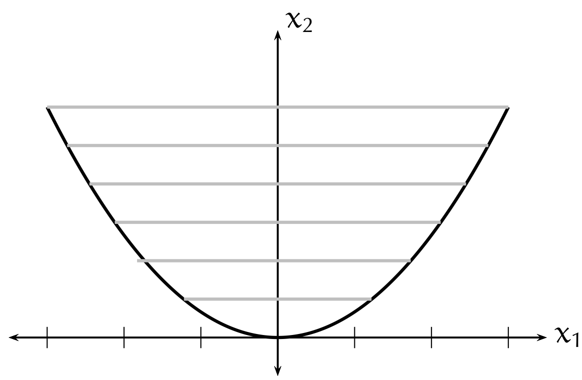

Figure 21.5.8: A quasiconvex function from $[-3, 3]$ to $\mathbb{R}$. Several lower contour sets are illustrated by the gray horizontal lines. If you project them into the x-axis, you get the actual lower contour sets, which are defined by $\{x : f(x) \leq y\}$ for $y = 0.5$, 1, 1.5, 2, 2.5, and 3. They are all convex, as expected for a quasiconvex function.

With a little thought, you should be able to see that the upper contour sets, which have the form $\{x : f(x) \geq y\}$ consist of two disjoint closed intervals for $0 < y \leq 3$.

21.5.9 Strict Quasiconvexity and Quasiconcavity

One consequence of Proposition 21.5.4 is that a function is quasiconcave if for every x, y with $f(x) \geq f(y)$ and every $t, 0 \leq t \leq 1$, we have $f\big((1-t)x + ty\big) \geq f(y)$. We can use this criterion for quasiconcavity to define *strict quasiconcavity*.

Strict Quasiconcavity and Strict Quasiconvexity. Let $f\colon S \to \mathbb{R}$ where S is a convex set.

1. The function f is *strictly quasiconcave* on S if and only if for every $x, y \in S$ with $f(x) \geq f(y)$ and every t with $0 < t < 1$, we have $f\big((1-t)x + ty\big) > f(y)$.
2. The function $f\colon S \to \mathbb{R}$ is *strictly quasiconvex* if and only if for every $x, y \in S$ with $f(x) \geq f(y)$ and every t with $0 < t < 1$, we have $f\big((1-t)x + ty\big) < f(y)$.

A strictly quasiconcave (quasiconvex) function can have at most one maximizer (minimizer).

Theorem 21.5.9. *Let $f\colon S \to \mathbb{R}$ where S is a convex set. Suppose f is strictly quasiconcave there are $x_0, x_1 \in S$ so that for all $y \in S$, $f(y) \leq f(x_i)$, $i = 0, 1$. Then $x_0 = x_1$.*

Proof. Since both x_i maximize f over S, $f(x_1) = f(x_0)$. If $x_0 \neq x_1$, define $x_2 = \frac{1}{2}x_0 + \frac{1}{2}x_1$. By strict quasiconcavity, $f(x_2) > f(x_0)$, showing that x_0 is not a maximum. This contradiction shows that $x_0 = x_1$. ∎

A similar result holds for minima of a strictly quasiconvex function.

21.5.10 Quasiconvexity is Not Ordinal Convexity

It's easy to find examples of quasiconvex functions that are not convex.

► Example 21.5.10: Quasiconvex does Not Mean Convex. Consider $f(x, y) = (x^2 + y^2)^{1/3}$. The function $g(x, y) = x^2 + y^2$ is convex (the Hessian is $2\mathbf{I}_2$). Raising it to the 1/3-power is an increasing transformation. Thus f is quasiconvex.

However, f is not convex. To see this, we compute $f(0, 0) = 0$ and $f(1, 1) = 2^{1/3}$. Then $f(t, t) = 2^{1/3}t^{2/3} > 2^{1/3}t = (1 - t)f(0, 0) + tf(1, 1)$ for $0 < t < 1$. If it were convex, the left-hand side would be no larger than the right. Since it is larger, it is not convex. ◄

Even transforming the function is not good enough. Not all quasiconcave functions are transformations of concave functions, nor are all quasiconvex functions transformations of convex functions.

Quasiconcave functions that **are** ordinally equiivalent to a concave function are called *concavifiable*. Similarly, a quasiconvex function that is ordinally equivalent to a convex function is called *convexifiable*. There are quasiconcave functions that are not concavifiable.[2]

[2] This has been known since Bruno de Finetti (1949), Sulle stratificazioni convesse, *Ann. Mat. Pura Applicata* **30**, 173–183. The problem of finding whether a given quasiconcave function is an increasing transformation of a concave function was posed (in terms of level sets) by Werner Fenchel (1953), *Convex Sets, Cones, and Functions,* lecture notes, Princton University, Department of Mathematics, Princeton, NJ. Although some results are known, it still does not have a completely satisfactory solution.

21.6 Quasiconvex/concave Support Property

There is a support property that characterizes quasiconvex and quasiconcave functions.

Support Property Theorem II. *Suppose* f *is* $\mathcal{C}^1$ *on* U*, an open convex subset of* $\mathbb{R}^m$*. The function* f *is quasiconcave if and only if for all* $\mathbf{x}, \mathbf{y} \in U$

$$f(\mathbf{y}) \geq f(\mathbf{x}) \textit{ implies } \big[Df(\mathbf{x})\big](\mathbf{y} - \mathbf{x}) \geq 0. \tag{21.6.10}$$

The function f *is quasiconvex if and only if for all* $\mathbf{x}, \mathbf{y} \in U$

$$f(\mathbf{y}) \geq f(\mathbf{x}) \textit{ implies } \big[Df(\mathbf{x})\big](\mathbf{y} - \mathbf{x}) \leq 0.$$

Proof (Only if). Here f is quasiconcave. Suppose $f(\mathbf{y}) \geq f(\mathbf{x})$. Consider $(1 - \varepsilon)\mathbf{x} + \varepsilon\mathbf{y} = \mathbf{x} + \varepsilon(\mathbf{y} - \mathbf{x})$ for $0 < \varepsilon < 1$. By quasiconcavity,

$$f\big(\mathbf{x} + \varepsilon(\mathbf{y} - \mathbf{x})\big) \geq f(\mathbf{x}) \quad \text{so} \quad \frac{f\big(\mathbf{x} + \varepsilon(\mathbf{y} - \mathbf{x})\big) - f(\mathbf{x})}{\varepsilon} \geq 0.$$

Let $\varepsilon \to 0$ to obtain $[Df(\mathbf{x})](\mathbf{y} - \mathbf{x}) \geq 0$, which establishes the result. ∎ (Only If)

Proof continues . . .

21.6.1 Support Property Theorem II, Part II

Proof (If). Now suppose that for all $\mathbf{x}, \mathbf{y} \in U$; $f(\mathbf{y}) \geq f(\mathbf{x})$ implies $[Df(\mathbf{x})](\mathbf{y} - \mathbf{x}) \geq 0$. We must show that f is quasiconcave.

We do this by contradiction. If f is not quasiconcave, there are $\mathbf{x}_0, \mathbf{x}_1$ and $0 < t_0 < 1$ with $f(\mathbf{x}_1) \geq f(\mathbf{x}_0)$ and $f\big((1 - t_0)\mathbf{x}_0 + t_0\mathbf{x}_1\big) < f(\mathbf{x}_0)$. Define

$$\mathbf{x}_t = (1 - t)\mathbf{x}_0 + t\mathbf{x}_1.$$

We can then write $f(\mathbf{x}_{t_0}) < f(\mathbf{x}_0)$.

Take an interval $I = [t_1, t_2] \subset [0, 1]$ with $t_0 \in I$ and $f(\mathbf{x}_t) \leq f(\mathbf{x}_0)$ for all $t \in I$, and $f(\mathbf{x}_{t_1}) = f(\mathbf{x}_{t_2}) = f(\mathbf{x}_0)$.

Now if $t \in I$ with $0 < t < 1$, $f(\mathbf{x}_1) \geq f(\mathbf{x}_0) \geq f(\mathbf{x}_t)$. But then by equation (21.6.10),

$$\big[Df(\mathbf{x}_t)\big](\mathbf{x}_0 - \mathbf{x}_t) \geq 0 \quad \text{and} \quad \big[Df(\mathbf{x}_t)\big](\mathbf{x}_1 - \mathbf{x}_t) \geq 0. \tag{21.6.11}$$

Of course, $\mathbf{x}_0 - \mathbf{x}_t = t(\mathbf{x}_0 - \mathbf{x}_1)$ and $\mathbf{x}_1 - \mathbf{x}_t = (1 - t)(\mathbf{x}_1 - \mathbf{x}_0)$. We substitute into equation (21.6.11) to obtain

$$-t\big[Df(\mathbf{x}_t)\big](\mathbf{x}_1 - \mathbf{x}_0) \geq 0 \quad \text{and} \quad (1 - t)\big[Df(\mathbf{x}_t)\big](\mathbf{x}_1 - \mathbf{x}_0) \geq 0.$$

Since $-t < 0$ and $(1 - t) > 0$, we must have

$$\big[Df(\mathbf{x}_t)\big](\mathbf{x}_1 - \mathbf{x}_0) = 0 \tag{21.6.12}$$

for all $t \in I$, $0 < t < 1$.

Now $0 < f(\mathbf{x}_0) - f(\mathbf{x}_{t_0}) = f(\mathbf{x}_{t_1}) - f(\mathbf{x}_{t_0})$, so by the Mean Value Theorem, there is $t_3 \in (t_1, t_0)$ with

$$0 < f(\mathbf{x}_0) - f(\mathbf{x}_{t_0}) = \big[Df(\mathbf{x}_{t_3})\big](\mathbf{x}_{t_1} - \mathbf{x}_{t_0}) = (t_0 - t_1)\big[Df(\mathbf{x}_{t_3})\big](\mathbf{x}_1 - \mathbf{x}_0).$$

But this contradicts equation (21.6.12). That contradiction shows that f is quasiconcave. ∎

21.6.2 Maximization and Quasiconcavity

We can use Support Property II for f to show that any point solves a maximization problem.

Theorem 21.6.1. *Suppose* f *is quasiconcave and* $\mathcal{C}^1$ *on a convex open set* U *and that* $\mathbf{p} = Df(\bar{\mathbf{x}})$. *Then* $\bar{\mathbf{x}}$ *maximizes* $f(\mathbf{x})$ *under the constraints* $\mathbf{p}\cdot\mathbf{x} \le \mathbf{p}\cdot\bar{\mathbf{x}}$ *and* $\mathbf{x} \in U$.

Proof. We prove this by contradiction. If false, there is $\mathbf{x} \in U$ with $\mathbf{p}\cdot\mathbf{x} \le \mathbf{p}\cdot\bar{\mathbf{x}}$ and $f(\mathbf{x}) > f(\bar{\mathbf{x}})$. Then for $\varepsilon > 0$ small enough $\mathbf{x} - \varepsilon\mathbf{p} \in U$ (because U is open) and $f(\mathbf{x} - \varepsilon\mathbf{p}) > f(\bar{\mathbf{x}})$ (by continuity). By Support Property II, $\mathbf{p}\cdot(\mathbf{x} - \varepsilon\mathbf{p}) \ge \mathbf{p}\cdot\bar{\mathbf{x}}$. But then, $\mathbf{p}\cdot\mathbf{x} \ge \mathbf{p}\cdot\bar{\mathbf{x}} + \varepsilon\|\mathbf{p}\|^2 > \mathbf{p}\cdot\bar{\mathbf{x}}$. This contradiction proves the theorem. ∎

Since the constraint $\mathbf{x} \in U$ cannot bind when U is open, $\bar{\mathbf{x}}$ maximizes f over all $\mathbf{x}$ with $\mathbf{p}\cdot\mathbf{x} \le \mathbf{p}\cdot\bar{\mathbf{x}}$. By Theorem 19.2.5,

$$\mathbf{h}^T\left[D^2 f(\bar{\mathbf{x}})\right]\mathbf{h} \le 0$$

for all $\mathbf{h}$ obeying $\mathbf{p}\cdot\mathbf{h} = 0$ or equivalently the bordered Hessian

$$\mathbf{B} = \begin{bmatrix} 0 & \mathbf{p}^T \\ \mathbf{p} & D^2 f \end{bmatrix}$$

has bordered principal minors that obey $(-1)^{n-1}\mathbf{A}_n \ge 0$ for $n = 3, \dots, m+1$. This provides a second-derivative test for quasiconcavity.

21.6.3 Uniqueness of Supports

When f is differentiable, we can show that the derivative is the only vector (up to scalar multiplication) that can support the upper (or lower) contour set. More precisely, we say that $\mathbf{p}$ *supports a set* S at $\mathbf{x}_0$ if $\mathbf{p}\cdot\mathbf{x} \geq \mathbf{p}\cdot\mathbf{x}_0$ whenever $\mathbf{x} \in S$. We apply this idea to the upper contour set $\{\mathbf{x} : f(\mathbf{x}) \geq f(\mathbf{x}_0)\}$ at $\mathbf{x}_0$. It is important to realize that this result does not require quasiconcavity. However, if the function is not quasiconcave, the upper contour set may not have supports.

Theorem 21.6.2. *Suppose* f *is differentiable at* $\mathbf{x}_0$ *with* $Df(\mathbf{x}_0) \neq \mathbf{0}$ *and that* $\mathbf{p}$ *supports* $\{\mathbf{x} : f(\mathbf{x}) \geq f(\mathbf{x}_0)\}$ *at* $\mathbf{x}_0$. *Then* $\mathbf{p} = \alpha\, Df(\mathbf{x}_0)$ *for some* $\alpha \neq 0$.

21.6.4 Uniqueness of Supports: Proof

Proof. Let

$$z = Df - \frac{\mathbf{p} \cdot Df}{\|\mathbf{p}\|^2} \mathbf{p}.$$

Then $\mathbf{p} \cdot z = 0$ and

$$Df \cdot z = \|Df\|^2 - |\mathbf{p} \cdot Df|^2 / \|\mathbf{p}\|^2.$$

By the Cauchy-Schwarz inequality, either $\mathbf{p}$ is proportional to Df (and we are done) or $Df \cdot z > 0$.

In the case where $D \cdot fz > 0$, the first-order Taylor approximation $f(\mathbf{x}_0) + \alpha\, Df \cdot z > f(\mathbf{x}_0)$ shows

$$f(\mathbf{x}_0 + \alpha z) > f(\mathbf{x}_0)$$

for small $\alpha > 0$. Continuity of f then yields

$$f(\mathbf{x}_0 + \alpha z - \varepsilon \mathbf{p}) > f(\mathbf{x}_0)$$

for $\varepsilon > 0$ small. Apply Support Property II to obtain

$$\begin{aligned} \mathbf{p} \cdot \mathbf{x}_0 &\le \mathbf{p} \cdot (\mathbf{x}_0 + \alpha z - \varepsilon \mathbf{p}) \\ &= \mathbf{p} \cdot \mathbf{x}_0 - \varepsilon \|\mathbf{p}\|^2 \\ &< \mathbf{p} \cdot \mathbf{x}_0. \end{aligned}$$

This contradiction shows $Df \cdot z > 0$ is impossible. Therefore $\mathbf{p}$ is proportional to $Df(\mathbf{x}_0)$. ∎

21.6.5 Bordered Hessian Test for Quasiconcavity

Since $\lambda\mathbf{p} = Df$, we can instead use the bordered Hessian with Df. The bordered principal minors are multiplied by λ^2, so the signs are unchanged. This yields the following theorem.

Theorem 21.6.3. *Suppose $f\colon U \to \mathbb{R}$ is a $\mathcal{C}^2$ function on some open convex set $U \subset \mathbb{R}^m$ with $m > 1$. Consider the bordered Hessian*

$$\mathbf{B} = \begin{bmatrix} 0 & Df \\ Df^T & D^2f \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_m} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_m} & \frac{\partial^2 f}{\partial x_1\,\partial x_m} & \cdots & \frac{\partial^2 f}{\partial x_m^2}. \end{bmatrix}$$

1. *If the bordered leading principal minors $\mathbf{B}_k$ obey $(-1)^{n-1}\mathbf{B}_n > 0$ on U for $n = 3, \ldots, m+1$, then f is quasiconcave on U.*
2. *If all non-trivial bordered leading principal minors are negative on U, then f is quasiconvex on U.*
3. *If f is quasiconcave on U, then every k^{th} order bordered principal minor $\tilde{\mathbf{B}}_k$ obeys $(-1)^{n-1}\tilde{\mathbf{B}}_n \geq 0$ on U for $n = 3, \ldots, m+1$.*
4. *If f is quasiconvex on U, then all non-trivial bordered principal minors are non-positive on U.*

Proof. For the quasiconcave case, see Arrow and Enthoven.[3] Applying the result to $-f$ yields the quasiconvex case. ∎

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[3] Arrow, Kenneth J., and Alain C. Enthoven (1961), Quasi-concave programming, *Econometrica* 29, 779–800.