23. Eigenvalues and Eigenvectors

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Eigenvalues and eigenvectors have a variety of uses. They allow us to solve linear difference and differential equations. For many non-linear equations, they inform us about the long-run behavior of the system. They are also useful for defining functions of matrices.

23.1 Eigenvalues

We start with eigenvalues.

Eigenvalues and Spectrum. Let $\mathbf{A}$ be an $m \times m$ matrix. An *eigenvalue* (*characteristic value*, *proper value*) of $\mathbf{A}$ is a number λ so that $\mathbf{A} - \lambda\mathbf{I}$ is singular.

The *spectrum of* $\mathbf{A}$, $\sigma(\mathbf{A}) = \{\text{eigenvalues of } \mathbf{A}\}$.

Sometimes it's possible to find eigenvalues by inspection of the matrix.

► **Example 23.1.1: Some Eigenvalues.** Suppose

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Here it is pretty obvious that subtracting either $\mathbf{I}$ or $2\mathbf{I}$ from $\mathbf{A}$ yields a singular matrix.

As for matrix $\mathbf{B}$, notice that subtracting $\mathbf{I}$ leaves us with two identical columns (and rows), so 1 is an eigenvalue. Less obvious is the fact that subtracting $3\mathbf{I}$ leaves us with linearly independent columns (and rows), so 3 is also an eigenvalue.

We'll see in a moment that 2×2 matrices have at most two eigenvalues, so we have determined the spectrum of each: $\sigma(\mathbf{A}) = \{1, 2\}$ and $\sigma(\mathbf{B}) = \{1, 3\}$. ◄

23.2 Finding Eigenvalues: 2×2

We have several ways to determine whether a matrix is singular. One method is to check the determinant. It is zero if and only the matrix is singular. That means we can find the eigenvalues by solving the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{23.2.1}$$

When $\mathbf{A}$ is $m \times m$, the left hand side of equation (23.2.1) is a polynomial in λ, called the *characteristic polynomial*.

When $\mathbf{A}$ is a 2×2 matrix, we have

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) \\ &= \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \qquad (23.2.2) \\ &= p(\lambda) \end{aligned}$$

Here the characteristic equation is a polynomial of degree two, $p(\lambda)$. It has at most two distinct roots.

23.3 Finding Eigenvalues: $m \times m$

Let's consider the eigenvalues of a general $m \times m$ matrix. The characteristic equation is

$$\begin{aligned}
0 &= \det(\mathbf{A} - \lambda\mathbf{I}) \\
&= \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} - \lambda \end{vmatrix} \\
&= (-1)^m \big(\lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m\big) \\
&= p(\lambda)
\end{aligned}$$

for some numbers $a_1, \ldots, a_m$. The characteristic polynomial $p(\lambda)$ is a polynomial of degree m.

In general, the characteristic equation of an $m \times m$ matrix is a polynomial of degree m, with at most m distinct roots. The spectrum of an $m \times m$ matrix has at most m elements.

In fact, the Fundamental Theorem of Algebra states that the characteristic polynomial $p(\lambda)$ can be factored as

$$p(\lambda) = (-1)^m(\lambda - \lambda_1)\cdots(\lambda - \lambda_m) \tag{23.3.3}$$

where the λ_i are the (possibly complex) roots of the polynomial, the eigenvalues. They need not be distinct, so the spectrum may include fewer than m points.

$\mathbf{A}$ and the Characteristic Polynomial. We can also apply the characteristic polynomial $p(\lambda)$ to the matrix $\mathbf{A}$.[1] Then

$$\begin{aligned}
p(\mathbf{A}) &= (-1)^m\big(\mathbf{A}^m + a_1\mathbf{A}^{m-1} + \cdots + a_{m-1}\mathbf{A} + a_m\mathbf{I}\big) \\
&= (-1)^m(\mathbf{A} - \lambda_1\mathbf{I})\cdots(\mathbf{A} - \lambda_m\mathbf{I}) \\
&= \mathbf{0}.
\end{aligned}$$

[1] This is not a proof, which is more involved.

23.4 Invariance of the Characteristic Polynomial

One interesting thing about the characteristic polynomial is that it is the same regardless of the basis used to write the matrix. That's another way of saying that the characteristic polynomial really belongs the transformation represented by the matrix.

Recall our change of basis formula for matrices representing linear transformations in equation (31.11.1). Given a new basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ with basis matrix $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_m]$, the matrix in the new coordinates becomes

$$\mathbf{A}_{\mathcal{B}} = \mathbf{B}^{-1}\mathbf{A}_{\mathcal{E}}\mathbf{B}. \tag{31.11.1}$$

Now

$$\begin{aligned}
\det(\mathbf{A}_{\mathcal{B}} - \lambda\mathbf{I}) &= \det(\mathbf{B}^{-1}\mathbf{A}_{\mathcal{E}}\mathbf{B} - \lambda\mathbf{I}) \\
&= \det(\mathbf{B}^{-1}\mathbf{A}_{\mathcal{E}}\mathbf{B} - \lambda\mathbf{B}^{-1}\mathbf{I}\mathbf{B}) \\
&= \det \mathbf{B}^{-1}(\mathbf{A}_{\mathcal{E}} - \lambda\mathbf{I})\mathbf{B} \\
&= (\det \mathbf{B}^{-1})\big(\det(\mathbf{A}_{\mathcal{E}} - \lambda\mathbf{I})\big)(\det \mathbf{B}) \\
&= \det(\mathbf{A}_{\mathcal{E}} - \lambda\mathbf{I}).
\end{aligned}$$

which shows that the characteristic polynomial of a transformation is the same in all bases.

23.5 Coefficients of the Characteristic Polynomial

Since the characteristic polynomial is the same in all bases, its coefficients are the same in all bases too. Such numbers are referred to as *invariants*.

We can rewrite equation (23.2.2) to expose two invariants.

$$0 = \lambda^2 - \lambda \operatorname{tr} \boldsymbol{A} + \det \boldsymbol{A}$$

where $\operatorname{tr} \boldsymbol{A}$ is the *trace*, the sum of the diagonal elements, $\operatorname{tr} \boldsymbol{A} = \sum_i a_{ii}$. Both the trace and determinant are independent of the basis used. The coefficient on the λ^{m-1} term is always $(-1)^{m-1} \operatorname{tr} \boldsymbol{A}$ and the constant term is always $\det \boldsymbol{A}$.

We can use equation (23.5.4) to write the trace and determinant in terms of eigenvalues.

$$\begin{aligned} p(\lambda) &= (-1)^m (\lambda - \lambda_1) \cdots (\lambda - \lambda_m) \qquad (23.5.4) \\ &= (-1)^m \left(\lambda^m - \lambda \left(\sum_{i=1}^{m} \lambda_i \right) + \cdots + (-1)^m \left(\prod_{i=1}^{m} \lambda_i \right) \right). \end{aligned}$$

It follows that

$$\operatorname{tr} \boldsymbol{A} = \lambda_1 + \cdots + \lambda_m \quad \text{and} \quad \det \boldsymbol{A} = \lambda_1 \cdots \lambda_m.$$

23.6 Finding Eigenvectors

For each eigenvalue λ there is an *eigenspace*. It is

$$\ker(\mathbf{A} - \lambda\mathbf{I}) = \{\boldsymbol{v} \in \mathbb{R}^m : (\mathbf{A} - \lambda\mathbf{I})\boldsymbol{v} = \mathbf{0}\} = \{\boldsymbol{v} \in \mathbb{R}^m : \mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v}\}.$$

The non-zero elements of the eigenspace are called *eigenvectors*.

We can find elements of the eigenspace for eigenvalue λ by solving the equation

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{v} = \mathbf{0} \quad \text{or} \quad \mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v}$$

▶ **Example 23.6.1: Some Eigenvectors.** Consider the matrices from Example 23.1.1.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

We set

$$\mathbf{0} = (\mathbf{A} - \mathbf{I})\boldsymbol{v} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}.$$

Since $v_2 = 0$, the eigenspace consists of vectors of the form $(v_1, 0)$ for $\lambda = 1$. Now for $\lambda = 2$, we set

$$\mathbf{0} = (\mathbf{A} - 2\mathbf{I})\boldsymbol{v} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ 0 \end{pmatrix}.$$

Now $v_1 = 0$, so vectors of the form $(0, v_2)$ comprise the eigenspace for $\lambda = 2$.

As for $\mathbf{B}$, $\sigma(\mathbf{B}) = \{1, 3\}$ so we examine

$$\mathbf{0} = (\mathbf{B} - \mathbf{I})\boldsymbol{v} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 \\ v_1 + v_2 \end{pmatrix}$$

Here $v_2 = -v_1$ so eigenvectors for $\lambda = 1$ have the form $v_1(1, -1)$. As for $\lambda = 3$,

$$\mathbf{0} = (\mathbf{B} - 3\mathbf{I})\boldsymbol{v} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_1 + v_2 \\ v_1 - v_2 \end{pmatrix}$$

and the eigenvectors have the form $v_1(1, 1)$. ◀

23.7 Complex Eigenvectors

Even when we are dealing with a real matrix, the eigenvalues and eigenvectors can be complex.

Consider

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{which obeys} \quad \mathbf{A}^2 = -\mathbf{I}.$$

The characteristic polynomial is

$$p(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = \mathbf{A} = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$

It follows that the eigenvalues are purely imaginary numbers, $\sigma(\mathbf{A}) = \{+i, -i\}$.

To find the corresponding eigenvectors, we first set

$$(\mathbf{A} - i\mathbf{I})\mathbf{u} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \mathbf{u} = \begin{pmatrix} -iu_1 + u_2 \\ -u_1 - iu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with solution $\mathbf{u} = (i, 1)$.

Then set

$$(\mathbf{A} + i\mathbf{I})\mathbf{u} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \mathbf{u} = \begin{pmatrix} iv_1 + v_2 \\ -v_1 + iv_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with solution $\mathbf{v} = (-i, 1)$.

Notice that we found eigenvectors that are complex conjugates of one another. It is always possible to do this, although it is not necessary because we can always multiply one of them by any non-zero complex number to get another eigenvector.

Theorem 23.7.1. *Suppose a matrix $\mathbf{A}$ has real entries. If λ_1 is a complex eigenvalue, then so is $\overline{\lambda_1}$. If $\mathbf{u}$ is a eigenvector corresponding to λ_1, then $\overline{\mathbf{u}}$ is an eigenvector corresponding to $\overline{\lambda_1}$.*

Proof. Since $\mathbf{A}$ has real entries, the characteristic polynomial $p(\lambda)$ has real coefficients. Then

$$\overline{p(\lambda_1)} = 0 = p(\overline{\lambda_1})$$

showing that $\overline{\lambda_1}$ is an eigenvalue if λ_1 is an eigenvalue.

Now suppose $\mathbf{u}$ is an eigenvector for λ_1. Then

$$\mathbf{A}\mathbf{u} = \lambda_1 \mathbf{u}.$$

It follows that

$$\begin{aligned} \overline{\mathbf{A}\mathbf{u}} &= \overline{\lambda_1 \mathbf{u}} && \text{by conjugation} \\ \mathbf{A}\overline{\mathbf{u}} &= \overline{\lambda_1}\,\overline{\mathbf{u}} && \text{since } \mathbf{A} \text{ is real} \end{aligned}$$

showing that $\overline{\mathbf{u}}$ is a eigenvector for $\overline{\lambda_1}$. ∎

23.8 The Leslie Population Model

Simon and Blume employ a form of Leslie population model to show how eigenvalues and eigenvectors can be used. We too will examine that, but do things a bit differently to highlight the role of eigenvalues and eigenvectors.

This simple version of the Leslie model involves an organism that lies for exactly two years. Let b_i be the birth rate to individuals in their i^{th} year, $i = 1, 2$, and d_i the death rate in each year. Survival in each year is $(1 - d_i)$ and the fact that the organism lives at most two years means that $d_2 = 1$. Let x_n be the number of individuals in their first year in year n and y_n be the number in their second (and final) year in year n.

The following difference equations describe the evolution of the system.

$$\begin{aligned} x_{n+1} &= b_1 x_n + b_2 y_n \\ y_{n+1} &= (1 - d_1)x_n + (1 - d_2)y_n = (1 - d_1)x_n. \end{aligned}$$

since $d_2 = 1$. Write this in matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ 1 - d_1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Consider the case $b_1 = 1$, $b_2 = 4$, $d_1 = 1/2$, and of course $d_2 = 1$. The system becomes

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{A} \begin{pmatrix} x_n \\ y_n \end{pmatrix}. \tag{23.8.5}$$

Notice that the solution to this dynamical system is already clear. Given initial values (x_0, y_0), we find

$$\begin{aligned} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= \mathbf{A} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \mathbf{A} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \mathbf{A}^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &\cdots \\ \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \mathbf{A} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} = \mathbf{A} \left(\mathbf{A}^{n-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right) = \mathbf{A}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \end{aligned}$$

These matrix powers are not necessarily easy to calculate. At this point Simon and Blume start discussing a change of coordinates—to coordinates that will make the calculation easier.

But where did their new coordinates come from?

Eigenvectors!

23.9 Leslie Example: Eigenvectors and Coordinates

Eigenvectors are the key to solving this Simon and Blume mystery.

We start by finding the eigenvalues for the matrix in equation (23.8.5). The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 4 \\ 1/2 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2) = 0.$$

The eigenvalues are $\{+2, -1\}$.

Now we solve for the eigenvectors:

$$\begin{pmatrix} -1 & 4 \\ 1/2 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 4 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The general solutions are

$$u_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

We set $u_2 = v_2 = 1$. Then eigenvector $\mathbf{u}$ corresponds to $\lambda = 2$ while eigenvector $\mathbf{v}$ corresponds to $\lambda = -1$. These vectors are linearly independent, and so form a basis $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$.

We form the basis matrix

$$\mathbf{B} = (\mathbf{u}, \mathbf{v}) = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}.$$

We write (x, y) for the standard coordinates and (X, Y) for the $\mathcal{B}$ coordinates. As usual, we get the standard coordinate by multiplying the basis matrix by the coordinates in $\mathcal{B}$. Then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{B} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 4X - 2Y \\ X + Y \end{pmatrix}.$$

which is equation (15) in Chapter 23 of Simon and Blume. What about their mysterious equation (14)? We calculate $\mathbf{B}^{-1}$.

$$\mathbf{B}^{-1} = \begin{pmatrix} 1/6 & 1/3 \\ -1/6 & 2/3 \end{pmatrix}$$

when of course their equation (14) comes from

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{B}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/6 & 1/3 \\ -1/6 & 2/3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{6}x + \frac{1}{3}y \\ -\frac{1}{6}x + \frac{2}{3}y \end{pmatrix}.$$

23.10 Leslie Example: Dynamics

Now we rewrite the difference equation (23.8.5) in terms of X and Y

$$\begin{aligned}\begin{pmatrix}X_{n+1}\\Y_{n+1}\end{pmatrix} &= \mathbf{B}^{-1}\begin{pmatrix}x_{n+1}\\y_{n+1}\end{pmatrix}\\ &= \mathbf{B}^{-1}\mathbf{A}\begin{pmatrix}x_n\\y_n\end{pmatrix}\\ &= \mathbf{B}^{-1}\begin{pmatrix}1 & 4\\1/2 & 0\end{pmatrix}\begin{pmatrix}x_n\\y_n\end{pmatrix}\\ &= \mathbf{B}^{-1}\begin{pmatrix}1 & 4\\1/2 & 0\end{pmatrix}\mathbf{B}\begin{pmatrix}X_n\\Y_n\end{pmatrix}\end{aligned}$$

Rather than multiply out the matrix product, we reason our way through it. We think in terms of the underlying linear transformation and its action on the eigenvectors. It multiplies them by the corresponding eigenvalue.

Set $(X, Y) = (1, 0)$. Multiplying by $\mathbf{B}$ gives us the first eigenvector in standard coordinates. The dynamic matrix $\mathbf{A}$ doubles it. Then $\mathbf{B}^{-1}$ translates it back to $\mathcal{B}$ (eigenvector) coordinates, yielding $(2, 0)$. Doing this to $(0, 1)$ it gets multiplied by -1, yielding $(0, -1)$. It follows that

$$\mathbf{B}^{-1}\begin{pmatrix}1 & 4\\1/2 & 0\end{pmatrix}\mathbf{B}\begin{pmatrix}X_n\\Y_n\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}X_n\\Y_n\end{pmatrix} = \begin{pmatrix}2X_n\\-Y_n\end{pmatrix}.$$

Rewriting our dynamical system in terms of eigenvectors has given it the simple form

$$\begin{pmatrix}X_{n+1}\\Y_{n+1}\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}X_n\\Y_n\end{pmatrix} = \begin{pmatrix}2X_n\\-Y_n\end{pmatrix} \tag{23.10.6}$$

23.11 Leslie Example: The Solution

Equation (23.10.6) has solution

$$\begin{aligned}\begin{pmatrix} X_n \\ Y_n \end{pmatrix} &= \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^n \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} \\ &= \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} \\ &= \begin{pmatrix} 2^n X_0 \\ (-1)^n Y_0 \end{pmatrix}.\end{aligned}$$

That is not quite what we want as it is in the eigenvector coordinates. We would prefer the original coordinates. But that just involves application of $\mathbf{B}$ and $\mathbf{B}^{-1}$.

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{B} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \mathbf{B} \begin{pmatrix} 2^n X_0 \\ (-1)^n Y_0 \end{pmatrix} = \begin{pmatrix} 2^n(4X_0) + (-1)^{n+1}(2Y_0) \\ 2^n X_0 + (-1)^n Y_0 \end{pmatrix}.$$

and

$$\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \mathbf{B}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{6}x_0 + \frac{1}{3}y_0 \\ -\frac{1}{6}x_0 + \frac{2}{3}y_0 \end{pmatrix}$$

which means that in standard coordinates

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 2^n(\frac{2}{3}x_0 + \frac{4}{3}y_0) + (-1)^n(-\frac{1}{3}x_0 + \frac{4}{3}y_0) \\ 2^n(\frac{1}{6}x_0 + \frac{1}{3}y_0) + (-1^{)}n(-\frac{1}{6}x_0 + \frac{2}{3}y_0) \end{pmatrix}.$$

23.12 Leslie Example: Dynamics

Now that we have the general form of the solution, we can think about the long-run behavior of the system. This is best done using the eigenvector coordinates where

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} 2^n X_0 \\ (-1)^n Y_0 \end{pmatrix}.$$

If $X_0 = 0$, meaning that $x_0 = 4y_0$, $X_n = 0$ and the solution oscillates back and forth between Y_0 and $-Y_0$. If $Y_0 = 0$, meaning that $x_0 + 2y_0 = 0$, $Y_n = 0$ and the solution doubles every period.

Some of this translates into standard coordinates. If we consider $x_n/2^n$ and $y_n/2^n$, we find

$$x_n/2^n \to \frac{2}{3}x_0 + \frac{4}{3}y_0 \quad \text{and} \quad y_n/2^n \to \frac{1}{6}x_0 + \frac{1}{3}y_0.$$

The diagram illustrates how this works.

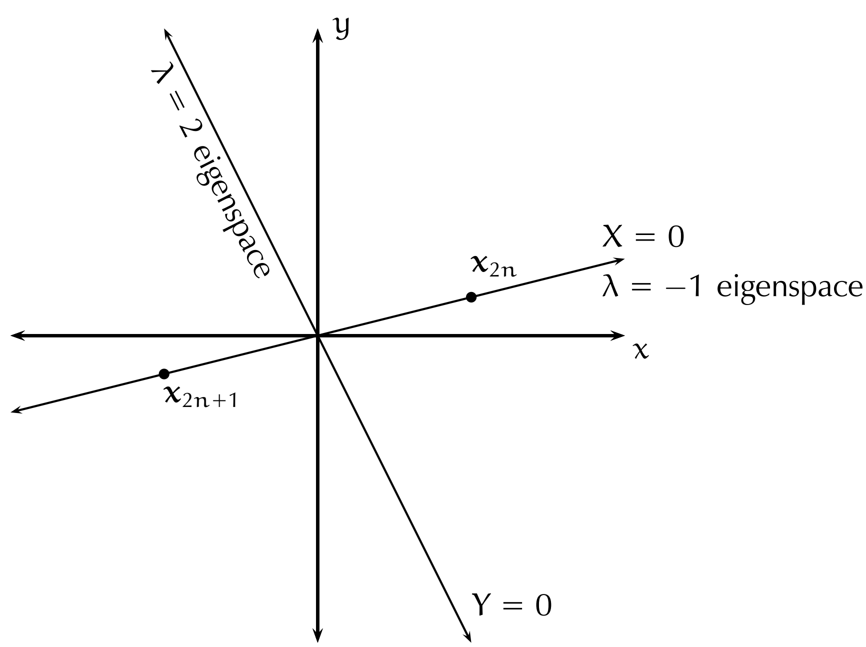

Figure 23.12.1: The $X = 0$ line shows the $\lambda = 2$ eigenspace while $Y = 0$ marks the $\lambda = -1$ eigenspace. If we start with a point on the $X = 0$ line, the solution is to oscillate back and forth, as between the points $\mathbf{x}_{2n}$ and $\mathbf{x}_{2n+1}$. A starting point on the $Y = 0$ line means that $\mathbf{x}_n = 2^n\mathbf{x}$. If we start at any point other than the origin, the motion will be a combination of these two factors, oscillates in the $X = 0$ direction as the points move out in the $Y = 0$ direction. In fact, every point $\mathbf{x}_n$ is one one of two parallel lines about the $Y = 0$ line.

23.13 General Systems of Difference Equations

We turn our attention to general difference equations of the form

$$z_{n+1} = \mathbf{A}z_n, \quad \mathbf{x}_0 \text{ given} \tag{23.13.7}$$

where $z_n \in \mathbb{R}^m$ and $\mathbf{A}$ is an $m \times m$ matrix. The Leslie model fits into this framework, and our method of solving our example Leslie model will apply to many of these systems.

The general method is to pick a coordinate system where $\mathbf{A}$ takes a nice form, such as a diagonal matrix $\mathbf{D}$. The point of using a diagonal matrix is that if

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix}, \quad \text{then} \quad \mathbf{D}^n = \begin{pmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_m^n \end{pmatrix}.$$

The omitted matrix entries are all zero.

Let $\mathbf{P}$ be the basis matrix and let $\mathbf{Z}$ indicate the vector z written in the new coordinates. Recall that the change of basis rule for vectors is

$$z = \mathbf{PZ}, \quad \text{or} \quad \mathbf{Z} = \mathbf{P}^{-1}z$$

If successful, the change of basis rules for matrices becomes

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}.$$

Now consider the vector with new coordinates $\mathbf{E}_i = (\delta_{ji})_{j=1}^m$. Then $\mathbf{DE}_i = \lambda_i \mathbf{E}_i$. It is an eigenvector of the linear transformation represented by $\mathbf{D}$ in the new coordinates. That means that $\mathbf{PE}_i$ it is an eigenvector of $\mathbf{A}$ in the standard coordinates.

The key to rewriting the difference equation (23.13.7) is to find the eigenvalues and eigenvectors of $\mathbf{A}$, exactly the method we used on the Leslie example.

23.14 Distinct Eigenvalues

Recall that we can write the characteristic polynomial of $\boldsymbol{A}$ as

$$p(\lambda) = (-1)^m(\lambda - \lambda_1)\cdots(\lambda - \lambda_m). \tag{23.5.4}$$

We will start by considering the case where all eigenvalues are distinct.

Theorem 23.14.1. *Let $\mathbf{A}$ be an $m \times m$ matrix. Suppose $\lambda_1, \dots, \lambda_k$ are any distinct eigenvalues of $\mathbf{A}$. Then the corresponding eigenvectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ are linearly independent.*

Proof. We prove this by induction. There's nothing to show for $k = 1$, so we start with $k = 2$. **Suppose they are not linearly independent**. Then there are non-zero β_1, β_2 with

$$\beta_1\boldsymbol{v}_1 + \beta_2\boldsymbol{v}_2 = \mathbf{0} \tag{23.14.8}$$

Applying $\boldsymbol{A}$, we find

$$\lambda_1\beta_1\boldsymbol{v}_1 + \lambda_2\beta_2\boldsymbol{v}_2 = \mathbf{0}$$

Now suppose $\lambda_2 = 0$, so $\lambda_1\beta_1\boldsymbol{v}_1 = \mathbf{0}$. Then $\lambda_1 \neq 0$ because the eigenvalues are distinct and β_1 is not zero, so we can conclude $\boldsymbol{v}_1 = \mathbf{0}$. But **this is impossible** as $\boldsymbol{v}_1$ is an eigenvector. This means $\lambda_2 \neq 0$.

Now that $\lambda_2 \neq 0$, we can divide by λ_2 to obtain

$$\frac{\lambda_1}{\lambda_2}\beta_1\boldsymbol{v}_1 + \beta_2\boldsymbol{v}_2 = \mathbf{0} \tag{23.14.9}$$

Subtracting equation (23.14.9) from (23.14.8), we find

$$\beta_1\boldsymbol{v}_1 = \beta_1\frac{\lambda_1}{\lambda_2}\boldsymbol{v}_1$$

Now $\beta_1 \neq 0$, so $\lambda_1 = \lambda_2$, **contradicting the fact that the λ_i are distinct**. This contradiction shows $\beta_1 = 0$, so $\beta_2 = 0$. The eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent.

Proof Continues...

23.15 Distinct Eigenvalues II

Proof (induction step). We next prove the induction step. Now suppose that ℓ of the eigenvectors are linearly independent with $\ell < k$. We will show that $\ell + 1$ of the eigenvectors are linearly independent.

Suppose not: Then there are β_i, not all zero, with

$$\sum_{i=1}^{\ell+1} \beta_i \boldsymbol{v}_i = \mathbf{0} \tag{23.15.10}$$

we may label the eigenvalues so that $\beta_{\ell+1} \neq 0$. Then apply $\boldsymbol{A}$ obtaining

$$\sum_{i=1}^{\ell+1} \lambda_i \beta_i \boldsymbol{v}_i = \mathbf{0} \tag{23.15.11}$$

Now suppose $\lambda_{\ell+1} = 0$, the other eigenvalues are not zero. In that case the remaining ℓ eigenvectors are linearly dependent, **contradicting the hypothesis**. It follows that $\lambda_{\ell+1} \neq 0$.

We may now divide equation (23.15.10) by $\beta_{\ell+1}$ and equation (23.15.11) by $\beta_{\ell+1}\lambda_{\ell+1}$ and subtract, obtaining

$$\sum_{i=2}^{\ell} \left(\frac{\beta_i}{\beta_{\ell+1}} - \frac{\beta_i}{\beta_{\ell+1}} \frac{\lambda_i}{\lambda_{\ell+1}} \right) \boldsymbol{v}_i = \mathbf{0}$$

But since the $\boldsymbol{v}_i$ for $i = 1, \ldots, \ell$ are linearly independent, $\beta_i - \lambda_i \beta_i / \lambda_{\ell+1} = 0$ for all i. By the contradiction hypothesis, there is at least one i with $\beta_i \neq 0$, implying $\lambda_i = \lambda_{\ell+1}$. This This **contradicts** the fact that the eigenvalues are distinct. It follows that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\ell+1}$ are linearly independent.

By induction, we conclude that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly independent. ∎

23.16 Distinct Eigenvalues III

Suppose $\mathbf{A}$ is an $\times m$ matrix. If m eigenvalues of $\mathbf{A}$ are distinct, their corresponding eigenvectors are linearly independent and as there are m of them, must form a basis for $\mathbb{R}^m$. Let

$$\mathbf{P} = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_m \end{pmatrix}$$

be the basis matrix for $\mathcal{P} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$. Then $\mathbf{E}_i$ gives the coordinates of the i^{th} eigenvector $\boldsymbol{v}_i$. Now $\mathbf{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$, which has coordinates $\lambda_i \mathbf{E}_i$ in $\mathcal{P}$. It follows that the matrix for the transformation defined by $\mathbf{A}$ is diagonal in the $\mathcal{P}$ coordinates. Summing up, we have just proved

Theorem 23.16.1. *Let $\mathbf{A}$ be an $m \times m$ matrix. Suppose the eigenvalues of $\mathbf{A}$, $\lambda_1, \ldots, \lambda_m$ are all distinct. Then the corresponding eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ form a basis, $\mathcal{P}$. Let $\mathbf{P}$ be the basis matrix for $\mathcal{P}$. Then*

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix}.$$

23.17 What If There Aren't m Distinct Eigenvalues?

In that case we can combine the terms with the same eigenvalue. The characteristic polynomial then can be rewritten

$$p(\lambda) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{m_i} \tag{23.17.12}$$

with each integer $m_i > 0$ and $\sum_{i=1}^{k} m_i = m$. The eigenvalues have been relabelled to eliminate the gaps in numbering after removing the duplicates.

We refer to m_i as the *algebraic multiplicity* of the eigenvalue λ_i and to $\dim\ker(\mathbf{A} - \lambda_i \mathbf{I}) = \nu(\lambda_i)$ as the *geometric multiplicity* or *nullity* of λ_i. The fact that λ_i is an eigenvalue means there is at least one non-zero vector with $(\mathbf{A} - \lambda_i \mathbf{I})\boldsymbol{v} = \mathbf{0}$. The geometric multiplicity must always be at least one. We state the following theorem without proof.

Theorem 23.17.1. *Let $\mathbf{A}$ be an $m \times m$ matrix with characteristic polynomial*

$$p(\lambda) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{m_i}. \tag{23.17.12}$$

Then for every $i = 1, \ldots, k$, $1 \le \nu(\lambda_i) \le m_i$.

Suppose each $\nu(\lambda_i) = m_i$. Then take a basis for each $\ker(\mathbf{A} - \lambda_i \mathbf{I})$ consisting of m_i elements. We use these as eigenvectors for λ_i. In this new basis, the transformation defined by $\mathbf{A}$ takes a block diagonal form

$$\begin{pmatrix} \mathbf{D}_1 & & & \\ & \mathbf{D}_2 & & \\ & & \ddots & \\ & & & \mathbf{D}_k \end{pmatrix}.$$

where the block $\mathbf{D}_i$ is the $m_i \times m_i$ diagonal matrix

$$\mathbf{D}_i = \begin{pmatrix} \lambda_i & & & \\ & \lambda_i & & \\ & & \ddots & \\ & & & \lambda_i \end{pmatrix}$$

23.18 Powers of Matrices

Suppose we want to write a general expression for the powers of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}.$$

The characteristic polynomial is $(1-\lambda)(2-\lambda)$, which has spectrum $\sigma(\mathbf{A}) = \{1, 2\}$. To find the eigenvectors, we solve the equations

$$\mathbf{0} = (\mathbf{A} - \mathbf{I})\boldsymbol{v}_1 = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \boldsymbol{v}_1 \quad \text{and} \quad \mathbf{0} = (\mathbf{A} - 2\mathbf{I})\boldsymbol{v}_2 = \begin{pmatrix} -1 & 0 \\ 3 & 0 \end{pmatrix} \boldsymbol{v}_2$$

which has solutions $\boldsymbol{v}_1 = (1, -3)$ and $\boldsymbol{v}_2 = (0, 1)$.

Let $\mathbf{P}$ be the basis matrix formed by the eigenvectors. Then

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}.$$

Now

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \mathbf{D}.$$

or

$$\mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{A}$$

Then

$$\begin{aligned} \mathbf{A}^n &= \overbrace{(\mathbf{P}\mathbf{D}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D}\mathbf{P}^{-1})\cdots(\mathbf{P}\mathbf{D}\mathbf{P}^{-1})}^{n \text{ times}} \\ &= \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}. \end{aligned}$$

It follows that

$$\mathbf{A}^n = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3(2^n - 1) & 2^n \end{pmatrix}.$$

23.19 Functional Calculus I

The *functional calculus* applies this formula to fractional powers, and any function f with $\sigma(\mathbf{A}) \subset \operatorname{dom} f$. Thus

$$f(\mathbf{A}) = \mathbf{P} f(\mathbf{D}) \mathbf{P}^{-1}$$

where

$$f(\mathbf{D}) = \begin{pmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_m) \end{pmatrix}$$

Applying the functional calculus to our matrix $\mathbf{A}$, we find its positive square root.

$$\sqrt{\mathbf{A}} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3(\sqrt{2} - 1) & \sqrt{2} \end{pmatrix}.$$

In fact, there are four square roots, found by taking $\pm\sqrt{\lambda_1} = \pm 1$ and $\pm\sqrt{\lambda_2} = \pm\sqrt{2}$ independently as the diagonal elements in $f(\mathbf{D})$.

Another square root is

$$\mathbf{P} \begin{pmatrix} -1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \mathbf{P}^{-1} = \begin{pmatrix} -1 & 0 \\ 3(\sqrt{2} + 1) & \sqrt{2} \end{pmatrix}.$$

23.20 Different Geometric and Algebraic Multiplicity

11/19/20

NB: The final is scheduled for Tuesday, December 8. It's officially at 5-7 pm, but in practice will be a take-home, like the midterms.

NB: I anticipate giving you one more homework assignment, probably on Tuesday.

We have one more case to consider. The geometric and algebraic multiplicity need not be the same. In that case, we don't have enough eigenvectors to span the space. An example of the problem is

$$\begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}.$$

The characteristic polynomial is

$$(5-\lambda)(3-\lambda)+1 = \lambda^2 - 8\lambda + 15 + 1 = (\lambda-4)^2.$$

The eigenvalue 4 has algebraic multiplicity 2. But when we solve $(\boldsymbol{A} - 4\mathbf{I})\boldsymbol{v} = 0$, we compute

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}$$

to find there is a one-dimensional family of solutions: $v_1(1,1)$. The geometric multiplicity is only one. We will use the eigenvector $\boldsymbol{v}_1 = (1,1)$.

23.21 Generalized Eigenvectors

How then do we find the rest of our basis? The solution is to use *generalized eigenvectors*.

Generalized Eigenvector. Let λ be an eigenvalue of the matrix $\mathbf{A}$. A non-zero vector $\boldsymbol{v}$ obeying $(\mathbf{A} - \lambda\mathbf{I})^{k-1}\boldsymbol{v} \neq \mathbf{0}$ with $(\mathbf{A} - \lambda\mathbf{I})^{k}\boldsymbol{v} = \mathbf{0}$ for some integer $k > 1$ is called a *generalized eigenvector* and k is its *rank*.

We require $k > 1$ because a rank one generalized eigenvector would be a regular eigenvector, with no generalization involved.

Notice that $p(\mathbf{A}) = (\mathbf{A} - 4\mathbf{I})^2$. We look for vectors solving $(\mathbf{A} - 4\mathbf{I})^2\boldsymbol{v} = \mathbf{0}$ but not solving $(\mathbf{A} - 4\mathbf{I})\boldsymbol{v} = \mathbf{0}$. That means that $(\mathbf{A} - 4\mathbf{I})\boldsymbol{v}_2$ is an eigenvector of $\mathbf{A}$. We happen to already have such an eigenvector available, $\boldsymbol{v}_1 = (1, 1)$. So we set

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (\mathbf{A} - 4\mathbf{I})\boldsymbol{v}_2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \boldsymbol{v}_2$$

Then $1 = v_1 - v_2$ and $1 = v_1 - v_2$. The solution is any vector of the form $\boldsymbol{v} = (v_1, v_1 - 1) = v_1\boldsymbol{v}_1 + (0, -1)$. We have already used $\boldsymbol{v}_1$ so we eliminate it by setting $v_1 = 0$, obtaining our second vector $\boldsymbol{v}_2 = (0, -1)$.

Now use $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ as a basis. Then

$$\mathbf{A}\boldsymbol{v}_1 = \begin{pmatrix} 5 - 1 \\ 1 + 3 \end{pmatrix} = 4\boldsymbol{v}_1 \quad \text{and} \quad \mathbf{A}\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \boldsymbol{v}_1 + 4\boldsymbol{v}_2.$$

It follows that the transformation given by $\mathbf{A}$ in the standard basis has matrix has the form

$$\mathbf{J} = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$$

in the new basis.

Given a generalized eigenvector $\boldsymbol{v}$ of rank k, the vectors $\boldsymbol{v}_k = \boldsymbol{v}$, $\boldsymbol{v}_{k-1} = (\mathbf{A} - \lambda\mathbf{I})\boldsymbol{v}_k$, $\ldots, \boldsymbol{v}_1 = (\mathbf{A} - \lambda\mathbf{I})^{k-1}\boldsymbol{v}_k$ are a *Jordan chain*. Of these vectors, only $\boldsymbol{v}_1 = (\mathbf{A} - \lambda\mathbf{I})^{k-1}\boldsymbol{v}_k$ is a true eigenvector. These generalized eigenvectors will appear in the order $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ in any *canonical basis* for $\mathbf{A}$.

Since there is only one Jordan chain for our matrix $\mathbf{A}$, we have found a canonical basis. The matrix $\mathbf{J}$ is in *Jordan canonical form*.

23.22 Example: Another Non-diagonalizable Matrix I

More complications can occur when the matrix is 3×3. The nullity of an eigenvalue can be one, two, or three, each with it's own type of Jordan canonical form.

We examine Problem 23.22. Here

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 0 \\ -1 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix}.$$

We start by computing the characteristic polynomial

$$p(\lambda) = (4 - \lambda)^3$$

Then $\lambda = 4$ is a root of algebraic multiplicity 3. We next search for eigenvectors.

$$(\mathbf{A} - 4\mathbf{I})\boldsymbol{v} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{v} = \mathbf{0}$$

This matrix has rank one, so its kernel has rank two by the Fundamental Theorem of Linear Algebra.

23.23 Example: Another Non-diagonalizable Matrix II

Rather than starting with the eigenvectors, we first search for the generalized eigenvectors. So we compute

$$(\mathbf{A} - 4\mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$$

The fact that $(\mathbf{A} - 4\mathbf{I})^2 = \mathbf{0}$, means that there are no generalized eigenvectors of rank three. This means there will be two eigenvectors, one with a corresponding generalized eigenvector of rank two.

We start by finding the generalized eigenvector $\boldsymbol{v}_3$. We solve by finding a vector $\boldsymbol{v}$ with $(\mathbf{A} - 4\mathbf{I})\boldsymbol{v} \neq \mathbf{0}$. Since $(\mathbf{A} - 4\mathbf{I})^2 = \mathbf{0}$, the solution will automatically generate a corresponding eigenvector. One easy solution is $\boldsymbol{v}_3 = (1, 0, 1)$ when $(\mathbf{A} - 4\mathbf{I}) = (0, 1, 0)$ which is our eigenvector $\boldsymbol{v}_2$. We need a second independent eigenvector and $\boldsymbol{v}_1 = (2, 0, 1)$ will do.

There are no generalized eigenvectors for $\boldsymbol{v}_1$ as

$$(\mathbf{A} - 4\mathbf{I})\boldsymbol{v}_1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \mathbf{0}.$$

We have two Jordan chains the trivial $(\boldsymbol{v}_1)$ and $(\boldsymbol{v}_2, \boldsymbol{v}_3)$. Putting the vectors in that order defines the basis matrix $\mathbf{P}$.

Then

$$\mathbf{P} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

and the Jordan form is the block diagonal matrix

$$\mathbf{J} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \left(\begin{array}{c|cc} 4 & 0 & 0 \\ \hline 0 & 4 & 1 \\ 0 & 0 & 4 \end{array}\right) = \begin{pmatrix} \mathbf{J}_1 & \\ & \mathbf{J}_2 \end{pmatrix}.$$

with

$$\mathbf{J}_1 = (4) \quad \text{and} \quad \mathbf{J}_2 = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}.$$

23.24 Jordan Canonical Form

We do not present a complete proof of the following theorem, but the preceding pages have shown you how to change bases to put a linear transformation into Jordan canonical form.

Jordan Canonical Form. *Let $\mathbf{A}$ be an $m \times m$ matrix with characteristic polynomial*

$$p(\lambda) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{m_i} \tag{23.17.12}$$

where m_i is the algebraic multiplicity of λ_i. Then there is a basis $\mathcal{P}$ with basis matrix $\mathbf{P}$ consisting of Jordan chains so that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J} = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$$

Where each block J_i has λ_i on the diagonal, zeros and ones on the superdiagonal, and zeros everywhere else. If $m_i > 1$, the block J_i may be decomposed into block diagonal matrices with ones on the superdiagonal corresponding to each generalized eigenvector of maximum rank and size given by that rank.

23.25 Some Possible Jordan Forms

To get a better feel for Jordan form, we present the possible Jordan forms when $\mathbf{A}$ is a 4×4 matrix and λ is an eigenvalue of algebraic multiplicity 4. There are 5 possibilities. The eigenvalue λ could have geometric multiplicity or nullity 4 (four blocks), nullity 3 (three blocks, one of the 2×2), nullity 2 (two blocks, either a pair of 2×2 or 1×1 and 3×3), and nullity 1 (all one block). Here's what they look like.

The nullity 4 and 3 cases are

$$\begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}.$$

In the first, the eigenspace is $\mathbb{R}^4$, so we can find a basis consisting of eigenvectors. In the second, the geometric multiplicity (nullity) is three, and we needed one generalized eigenvector. The other two vectors have no generalized eigenvector.

The two nullity 2 cases are

$$\begin{pmatrix} \lambda & & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix},$$

In the first, one eigenvector had no additional generalized eigenvectors, and the other one had two independent generalized eigenvectors. In the second, both had one independent generalized eigenvector.

Finally, when the nullity is one, there are three independent generalized eigenvectors, yielding:

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$$

23.26 Spectral Theorem

The Spectral Theorem gives sufficient conditions for a real or complex matrix to be diagonalizable, and necessary and sufficient conditions for a matrix to be diagonalizable via an orthonormal basis.

Normal Matrix. A matrix is *normal* if it commutes with its Hermitian conjugate, $\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^*$.

For real matrices, normal means that a matrix commutes with its transpose.

Spectral Theorem. *Suppose* $\mathbf{A}$ *is an* $m \times m$ *normal matrix. Then there is an orthonormal basis* $\{\mathbf{u}_1, \ldots \mathbf{u}_m\}$ *consisting of eigenvectors with basis matrix* $\mathbf{U}$*, necessarily obeying* $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}$ *such that* $\mathbf{D} = \mathbf{U}^*\mathbf{A}\mathbf{U}$ *is a diagonal matrix with diagonal elements consisting of the spectrum* $\sigma(\mathbf{A})$.

If a matrix is Hermitian, all of is eigenvalues are real.

Theorem 23.26.1. *If* $\mathbf{A}^* = \mathbf{A}$*, the elements of the spectrum are all real.*

Proof. To see this, if $\lambda \in \sigma(\mathbf{A})$, the Spectral Theorem yields a unit vector $\mathbf{u}$ with $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$. Now

$$
\begin{aligned}
\bar{\lambda}(\mathbf{u}\cdot\mathbf{u}) &= (\lambda\mathbf{u})\cdot\mathbf{u} \\
&= (\mathbf{A}\mathbf{u})\cdot\mathbf{u} = \mathbf{u}\cdot(\mathbf{A}^*\mathbf{u}) \\
&= \mathbf{u}\cdot(\mathbf{A}\mathbf{u}) = \mathbf{u}\cdot(\lambda\mathbf{u}) \\
&= \lambda(\mathbf{u}\cdot\mathbf{u}).
\end{aligned}
$$

Since $\mathbf{u}\cdot\mathbf{u} = 1$, $\lambda = \bar{\lambda}$, showing that λ is real. ∎

In particular, if $\mathbf{A}$ is a real symmetric matrix, $\mathbf{A}^* = \mathbf{A}^T = \mathbf{A}$, implying that every eigenvalue is real. In that case, the resulting basis vectors are also real, and $\mathbf{U}^* = \mathbf{U}^T$. Then $\mathbf{U}^T\mathbf{U} = \mathbf{I}$. This means that $\mathbf{U}$ is either a rotation or a rotation combined with a reflection on $\mathbb{R}^m$.

23.27 Quadratic Forms and Eigenvalues

One application of the Spectral theorem is to quadratic forms.

Theorem 23.27.1. *Suppose* $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ *for a symmetric real matrix* $\mathbf{A}$ *is a quadratic form on* $\mathbb{R}^m$.

(a) Q *is positive definite if and only if all of the eigenvalues of* $\mathbf{A}$ *are positive.*
(b) Q *is positive semidefinite if and only if all of the eigenvalues of* $\mathbf{A}$ *are non-negative.*
(c) Q *is negative definite if and only if all of the eigenvalues of* $\mathbf{A}$ *are negative.*
(d) Q *is negative semidefinite if and only if all of the eigenvalues of* $\mathbf{A}$ *are non-positive.*

Proof. By the Spectral Theorem, there is an orthonormal basis where $\mathbf{A}$ takes the diagonal form $\mathbf{D}$. Now

$$\begin{aligned} Q(\mathbf{x}) &= \mathbf{x}^T\mathbf{A}\mathbf{x} \\ &= (\mathbf{U}\mathbf{X})^T\mathbf{A}(\mathbf{U}\mathbf{X}) \\ &= (\mathbf{X}^T\mathbf{U}^T)\mathbf{A}(\mathbf{U}\mathbf{X}) \\ &= \mathbf{X}^T(\mathbf{U}^T\mathbf{A}\mathbf{U})\mathbf{X} \\ &= \mathbf{X}^T\mathbf{D}\mathbf{X} \end{aligned}$$

in the new basis.

We saw in section 16.8 that the Definite Matrices Theorem is equivalent to the conditions above when the matrix is diagonal. ∎

23.28 Functional Calculus II

A form of the functional calculus also applies to matrices in Jordan canonical form. But is rather more complex. We first consider the case of powers of $\mathbf{A}$. We start by examining powers of Jordan blocks.

Suppose

$$\mathbf{J} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{so} \quad \mathbf{J}^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 1 \\ 0 & \lambda^2 \end{pmatrix}.$$

Then

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}.$$

We can prove this by induction. It is true for $n = 1$. Now suppose it is true for n. Then

$$\mathbf{J}^{n+1} = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^{n+1} & \lambda^n + n\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix} = \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix}$$

showing it is true for $n + 1$. By induction, it is true for all $n = 1, 2, 3, \ldots$.

23.29 Powers of Jordan Blocks

What about a $k \times k$ Jordan block

$$J = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

Then we have powers of λ that successively populate the next superdiagonal until they are full at which point the matrix is upper triangular with the following form:

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} & \cdots & \frac{n!}{(n-k)!k!}\lambda^{n-k+1} \\ & \lambda^n & n\lambda^{n-1} & \cdots & \frac{n!}{(n-k+1)!(k-1)!}\lambda^{n-k+2} \\ & & \ddots & \ddots & \vdots \\ & & & \lambda^n & n\lambda^{n-1} \\ & & & & \lambda^n \end{pmatrix}$$

You will recognize the coefficients as the binomial coefficients

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

that occur in Taylor series.

Then $\mathbf{A}^n = \mathbf{P}^{-1}\mathbf{J}^n\mathbf{P}$.

We are now ready to consider power series of Jordan blocks.

23.30 Power Series of Jordan Blocks

Now suppose we have a power series that converges on the spectrum $\sigma(\mathbf{A})$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and form

$$f(\mathbf{J}) = \sum_{n=0}^{\infty} a_n \mathbf{J}^n.$$

On the diagonal, the terms are in $\mathbf{J}^n$ are λ^n, so we have

$$\sum_{n=0}^{\infty} a_n \lambda^n = f(\lambda).$$

The first superdiagonal has terms $n\lambda^{n-1}$, so we get

$$\sum_{n=0}^{\infty} a_n n \lambda^{n-1} = \sum_{n=0}^{\infty} a_n \frac{d}{d\lambda}\lambda^n = \frac{d}{d\lambda}\left(\sum_{n=0}^{\infty} a_n \lambda^n\right) = f'(\lambda),$$

assuming the power series of derivatives converges. On the k^{th} superdiagonal, we have

$$\sum_{n=0}^{\infty} a_n \frac{n!}{(n-k)!k!}\lambda^{n-k} = \sum_{n=0}^{\infty} \frac{a_n}{k!}\frac{d^k}{d\lambda^k}\lambda^n = \frac{1}{k!}\frac{d^k f}{d\lambda^k}(\lambda).$$

We conclude, that provided all of the relevant power series converge, if $\mathbf{J}$ is an $\ell \times \ell$ matrix, then $f(\mathbf{J})$ is a matrix with $f'(\lambda)$ on the diagonal and

$$\frac{1}{k!}\frac{d^k f}{d\lambda^k}(\lambda)$$

on the k^{th} superdiagonal for $k \leq \ell$.

More generally, we expect this pattern to hold for any f that is $\mathcal{C}^{\infty}$.

24. Ordinary Differential Equations: Scalar Equations

In many cases it is more convenient to model time continuously instead of discretely, as is done when using difference equations. Certain distinctions, such as that between stocks and flows, are more easily expressed in continuous time. Continuous time also us to use powerful methods from calculus and mathematical analysis that are not available in discrete time. Not everything is a benefit, and showing that a particular differential equation has a solution can lead to complications that have no analog in discrete time.

We start by converting a simple difference equation to continuous time. Consider the problem of determining the amount in a savings account at any time given a fixed interest rate r. In discrete time we can model this by assuming the interest is paid annually. If $y(t)$ is the amount in the account at time t, measured in years, we have

$$\frac{y(t+1) - y(t)}{y(t)} = r, \quad \text{or} \quad y(t+1) = (1+r)y(t).$$

24.1 Varying the Time Period

Although some accounts pay interest annually, many do not. Another common method is to pay interest monthly. In that case the annual interest rate in broken into 12 pieces. We shall approximate this by using an interest rate of $r/12$ per period, and measure time in units of $1/12^{th}$ year. Or maybe we prefer to consider daily compounding, breaking the interest rate into $r/365.25$ and using 365.25 periods per average year. Both (and more) can be accommodated by rewriting the equation as

$$\frac{y(t+\Delta t) - y(t)}{y(t)} = r \cdot \Delta t$$

where $\Delta t = 1/12$ or $1/365.25$ as needed.

Yet another method is to compound the interest continuously. For this we multiply both sides by $y/\Delta t$ to get

$$\frac{y(t+\Delta t) - y(t)}{\Delta t} = ry(t)$$

and let $\Delta t \to 0$. This yields the differential equation

$$\frac{dy}{dt}(t) = ry(t).$$

This is an example of an *ordinary differential equation*, because it expresses the derivative of a function (y) of a single variable (t) in terms of the function itself.

Of course, for the interest problem we expect to have one more piece of information, the amount y_0 in the account at some starting time t_0. The differential equation together with that initial data is called an *initial value problem*.

$$\frac{dy}{dt} = ry$$
$$y(t_0) = y_0.$$

We often take $t_0 = 0$.

24.2 Differential Equations: Another Example

Another example of a differential equation is $\dot{y} = y^2$ where $\dot{y}$ denotes dy/dt. You can verify that $y(t) = -1/t$ solves this equation, as does $y(t) = 1/(k - t)$. This type of solution doesn't work if $y(0) = 0$. In that case we have another solution: $y(t) = 0$.

More generally, if we have initial data $y(t_0) = y_0$, we set

$$y(t_0) = \frac{1}{k - t_0} = y_0.$$

Some manipulation shows

$$k = (1 + y_0 t_0)/y_0,$$

so

$$y(t) = \frac{y_0}{1 + y_0(t - t_0)}.$$

This solution works for any y_0 other than 0, showing that

$$y(t) = \begin{cases} 0 & \text{when } y_0 = 0 \\ \frac{y_0}{1+y_0(t-t_0)} & \text{otherwise} \end{cases}$$

is a general solution.

24.3 Ordinary Differential Equations

Ordinary differential equations also include equations such as

$$\dot{y} = t^2 y(t)$$

and equations involving higher derivatives such as $\ddot{y} = \dot{y} + y$ as well as simple equations involving only $\dot{y}$ and t as in

$$\dot{y} = t^3, \quad \dot{y} = e^t, \quad \text{and} \quad \dot{y} = \cos t + t.$$

These can be solved by simple integration with solutions

$$y(t) = t^4/4 + k, \quad y(t) = e^t + k, \quad \text{and} \quad y(t) = -\sin t + t^2/2 + k.$$

where k is a constant of integration.

Ordinary differential equations do not include anything involving partial derivatives. The fundamental division in the subject is between ordinary differential equations, based on ordinary derivatives and partial differential equations based on partial derivatives.

Ordinary Differential Equation. An ordinary differential equation is an equation involving a function y of a single variable t and at least one of y's derivatives. If the highest derivative is the k^{th} derivative, it is an k^{th} order equation. Such an equation can be explicit, with $y^{(k)}(t) = F(t, y, y', \dots, y^{(k-1)})$ or implicit with $0 = F(t, y, y', \dots, y^{(k)})$. If the equation does not explicitly include t it is called *autonomous*.

Sometimes a constant function will solve a differential equation. We refer to such solutions as *steady states*, *stationary solutions*, *equilibria*, and other such terms. We encountered such a solution when $y(t) = 0$, solved $\dot{y} = y^2$.

If y_0 is a steady state and $y(t) \to y_0$ as $t \to \infty$, we say that y_0 is *asymptotically stable*.

You'll notice that some of the solutions given above included a constant k. A *general solution* to a differential equation has one or more parameters such as our constant of integration k that can be adjusted to achieve any possible solution.

24.4 Solving Differential Equations: Integration

Some first order differential equations can be solved by integrating both sides. Thus $\dot{y} = ay$ can be rewritten $\dot{y}/y = a$ and integrated

$$\int \frac{dy}{y} = \int a\,dt \quad \text{or, more formally,} \quad \int \frac{1}{y}\frac{dy}{dt}\,dt = \int a\,dt.$$

so

$$\ln y = at + C, \quad \text{or} \quad y(t) = ke^{at}. \tag{24.4.1}$$

This is the general solution. If this is an initial value problem we can often pin down any constants of integration. So if $y(t_0) = y_0$, we substitute in equation (24.4.1) to find $y_0 = y(t_0) = ke^{at_0}$, implying $k = y_0e^{-at_0}$. Then substitute the expression for k back in (24.4.1), to obtain

$$y(t) = y_0e^{a(t-t_0)}.$$

Similarly, with some rearrangement, $\dot{y} = ay + b$ can be integrated when a and b are non-zero, obtaining the general solution

$$y(t) = -\frac{b}{a} + ke^{at}.$$

Again, if initial data is available, it can determine the value of k. So if

$$y_0 = y(t_0) = -\frac{b}{a} + ke^{at_0},$$

we find

$$k = e^{-at_0}\left(y_0 + \frac{b}{a}\right),$$

implying

$$y(t) = -\frac{b}{a} + \left(y_0 + \frac{b}{a}\right)e^{a(t-t_0)}.$$

24.5 Integrating Non-Autonomous Equations

Those were both autonomous equations. There are non-autonomous versions too, such as $\dot{y} = a(t)y$, with solution

$$y(t) = k \exp\left(\int^t a(s)\,ds\right)$$

of

$$y(t) = y_0 \exp\left(\int_{t_0}^t a(s)\,ds\right).$$

The symbol $\int^t$, with no lower limit indicates an indefinite integral (anti-derivative) using the variable t. Here s serves as a dummy variable inside the integral. Since it is confined to the integral, we can call it anything we want. However, t here is not recommended as it has the potential to cause confusion due to the fact it is not the same t as appears outside the integral. The t's outside the integral are the same as the t in the pseudo upper limit.

Other rather ordinary equations do not have closed form solutions. Often they can be numerically integrated to any desired degree of accuracy. In some cases, they are useful enough to be given a name. Thus $\int e^{-t^2}\,dt$ has no closed form expression, but with a factor of $2/\sqrt{\pi}$, it is called the *error function*, and gets wide use in statistics. Similarly, the Bessel functions are defined as solutions to Bessel's equation

$$t^2\ddot{y} + t\dot{y} + (t^2 - \alpha^2)y = 0$$

for any complex number α. Bessel functions have even had a major treatise written about them.[1]

[1] Watson, G.N., (1922) "A Treatise on the Theory of Bessel Functions", Cambridge University Press.

24.6 Integrating Factors I

11/24/20

NB: Problems 15 and 21 from Chapter 23 and problems 13 and 17 from Chapter 24 are due on Tuesday, December 1.

Equations such as

$$\dot{y} = a(t)y + b(t) \tag{24.6.2}$$

can be integrated by using an *integrating factor*. To see how this works, consider the simpler equation

$$\dot{y} = ay + b(t), \quad \text{or} \quad \dot{y} - ay = b(t). \tag{24.6.3}$$

Now

$$\frac{d}{dt}(e^{-at}y) = e^{-at}\dot{y} - ae^{-at}y = e^{-at}(\dot{y} - ay).$$

So if we use the form $\dot{y} - ay = b(t)$ and multiply by the integrating factor e^{-at}, we obtain

$$\frac{d}{dt}(e^{-at}y) = e^{-at}\big(\dot{y} - ay\big) = e^{-at}b(t)$$

which can now be integrated to obtain

$$e^{-at}y(t) = k + \int^t e^{-as}b(s)\,ds,$$

which we multiply by e^{at}, yielding

$$y(t) = ke^{at} + e^{at}\int^t e^{-as}b(s)\,ds.$$

24.7 Integrating Factors II

Now we apply the method of an integrating factor to our original equation $\dot{y} = a(t)y + b(t)$. Here the integrating factor is

$$A(t) = \exp\left(-\int^t a(s)\,ds\right)$$

so that we can move the $y(t)$ term in equation (24.6.2) to the left-hand side and multiply by $A(t)$ to obtain

$$\frac{d}{dt}\Big[A(t)y(t)\Big] = A(t)\Big[\dot{y}(t) - a(t)y(t)\Big] = A(t)b(t).$$

Then we integrate to get

$$A(t)y(t) = k + \left(\int^t A(s)b(s)\,ds\right)$$

yielding

$$y(t) = \exp\left(\int^t a(s)\,ds\right)\left[k + \left(\int^t e^{-\int^s a(u)\,du}\,b(s)\,ds\right)\right].$$

Notice that we have used two dummy variables. We use different letters for them to avoid any possibility of confusion.

24.8 Separable Differential Equations

A first order differential equation $\dot{y} = F(t, y)$ is *separable* if $F(t, y) = g(y)h(t)$ for some functions g and h. Then we have

$$\frac{dy}{dt} = g(y)h(t)$$

which can be written

$$\frac{dy}{g(y)} = h(t)\,dt.$$

Integrate the two sides to obtain

$$\int^{y} \frac{dy}{g(y)} = \int^{t} h(t)\,dt + C.$$

If there is an initial condition $y_0 = y(t_0)$ this can be used to set the limits of integration:

$$\int_{y_0}^{y} \frac{dy}{g(y)} = \int_{t_0}^{t} h(t)\,dt.$$

For example, if

$$\dot{y} = ty^2 \text{ and } y(0) = y_0,$$

we use

$$\int_{y_0}^{y} \frac{\dot{y}}{y^2}\,dy = \int_{0}^{t} t\,dt.$$

Then

$$-\frac{1}{y(t)} + \frac{1}{y_0} = \frac{t^2}{2}$$

which yields

$$y(t) = \frac{2y_0}{2 - t^2 y_0}.$$

If $y_0 > 0$, this solution has a singularity at $t = \pm\sqrt{2/y_0}$.

24.9 Example: Constant Relative Risk Aversion

► Example 24.9.1: Arrow-Pratt Relative Risk Aversion. Let $u(x)$ be a utility function The Arrow-Pratt measure of absolute risk aversion is

$$R_R(x) = -\frac{u''(x)x}{u'(x)}.$$

Suppose the relative risk aversion is constant, with $R_R(x) = b$. Then

$$u''(x) = -\frac{u'(x)b}{x}.$$

Substitute $v(x) = u'(x)$, so $u''(x) = v'(x)$. Then

$$\frac{dv}{dx} = -\frac{vb}{x}$$

which is separable in v and x. We rearrange the equation to separate them and integrate

$$\frac{dv}{v} = -b\frac{dx}{x} \quad \text{or} \quad \int \frac{dv}{v} = -b\int \frac{dx}{x},$$

with solution

$$\ln v = -b \ln x + C.$$

We take the exponential, obtaining

$$v = k_1 x^{-b}$$

where $k_1 = e^C$. We are not yet done. Now $u' = v = k_1 x^{-b}$. We integrate one more time to find

$$u = \int k_1 x^{-b} = \begin{cases} k_2 + k_1 \ln x & \text{if } b = 1, \\ k_2 + \dfrac{k_1}{1-b} x^{1-b} & \text{if } b \neq 1. \end{cases}$$

We normally assume $u' > 0$, which requires $k_1 > 0$. ◄

24.10 Example: Constant Absolute Risk Aversion

The previous page found the utility functions that have constant *relative* risk aversion. We now turn to the problem of finding utility functions with constant *absolute* risk aversion.

► Example 24.10.1: Arrow-Pratt Absolute Risk Aversion. Let $u(x)$ be a utility function The Arrow-Pratt measure of absolute risk aversion is

$$R_A(x) = -\frac{u''(x)}{u'(x)}.$$

Suppose the absolute risk aversion is constant, with $R_A(x) = b$. Then

$$bu'(x) = -u''(x).$$

Again substitute $v(x) = u'(x)$, so $u''(x) = v'(x)$. Then

$$bv = -\frac{dv}{dx}$$

This is separable, and we rewrite it as

$$-b\,dx = \frac{dv}{v}.$$

We integrate to obtain

$$-bx + C = \ln v,$$

so

$$v = e^{\ln v} = e^C e^{-bx} = k_1 e^{-bx}$$

Then, substitute $v = u'$, yielding

$$\frac{du}{dx} = k_1 e^{-bx} \quad \text{or} \quad du = k_1 e^{-bx}\,dx$$

Integrating again we obtain

$$u(x) = k_0 - k_2 e^{-bx}$$

where $k_2 = k_1/b > 0$ so that u is increasing in x. This the family of utility functions with constant absolute risk aversion. ◄

24.11 Example: Logistic Equation

▶ Example 24.11.1: Logistic Equation. Consider the *logistic equation*

$$\dot{y} = y(a - by)$$

where $a, b > 0$. This is clearly separable. We separate variables and integrate to write

$$\int^{y} \frac{dy}{y(a - by)} = \int^{t} dt.$$

The right-hand side is easy enough, but the left needs work. We use the method of partial fractions, which you might recall from calculus. We write

$$\frac{1}{y(a - by)} = \frac{A}{y} + \frac{B}{a - by}$$

and solve for the constants A and B. Then

$$\frac{1}{y(a - by)} = \frac{A(a - by)}{y(a - by)} + \frac{By}{y(a - by)}$$

so

$$1 = Aa - Aby + By = Aa + (B - Ab)y$$

which requires $Ab = B$ and $Aa = 1$. Substituting in the integral,

$$\begin{aligned}\int \frac{dy}{y(a - by)} &= \frac{1}{a}\int \frac{dy}{y} + \frac{b}{a}\int \frac{dy}{a - by} \\ &= \frac{1}{a}\ln y - \frac{1}{a}\ln(a - by) \\ &= \frac{1}{a}\ln \frac{y}{a - by}\end{aligned}$$

Thus

$$t + c = \frac{1}{a}\ln \frac{y}{a - by} \quad \text{or} \quad Ce^{at} = \frac{y}{a - by}$$

for some constant $C > 0$. Now solve for y to obtain

$$y = \frac{a}{b + ke^{-at}}.$$

where $a, b > 0$.

If $k = 0$ we have a steady state, $y = a/b$. For $k > 0$, $y(t) > 0$ and $y(t) \to a/b$.

The case $k < 0$ involves an extra complication as the denominator can be zero. Of course, if we have initial conditions, they will pin down k. ◀

24.12 Second Order Linear Equations

Second order linear ordinary differential equations frequently occur in economics. For example, they provide approximations to Euler equations for optimal and equilibrium growth models near a steady state or balanced growth path. The basic second order equation is one that is homogeneous with constant coefficients. It can be written

$$a\ddot{y} + b\dot{y} + cy = 0 \tag{24.12.4}$$

with $a, b, c \in \mathbb{R}$ and $\ddot{y} = d^2y/dt^2$.

We define a linear map from the vector space of $\mathcal{C}^2$ functions to vector space of $\mathcal{C}$ functions by

$$\mathbf{T}(y) = a\ddot{y} + b\dot{y} + cy$$

We can rewrite equation 24.12.4 as $\mathbf{T}(y) = 0$. The equation with a zero on the right hand side is called a *homogeneous equation*. If $\mathbf{T}(y)$ was equated to anything else, we would call it an *inhomogeneous equation*.

On important property of linear equations is that that if y_1 and y_2 are solutions to the homogeneous equation $\mathbf{T}(y) = 0$, so is any linear combination of y_1 and y_2. This is referred to as the *Principle of Superposition*. It applies to any linear map of the form

$$\mathbf{T}(y) = a_0\frac{d^n}{dt^n}y + a_1\frac{d^{n-1}}{dt^{n-1}}y + \cdots + a_{n-1}\dot{y} + a_n y$$

24.13 Solving Second Order Linear Equations

Equation (24.12.4) has solutions of the form e^{rt} for some constant r. We can find r by substituting $y = e^{rt}$ in equation (24.12.4):

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

yielding the *characteristic equation* (the name is not a coincidence)

$$ar^2 + br + c = 0. \tag{24.13.5}$$

By the quadratic formula, the solutions are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases to consider

1. $b^2 - 4ac > 0$, when there are two distinct real roots.
2. $b^2 - 4ac = 0$, when both roots are equal.
3. $b^2 - 4ac < 0$, when there are two complex conjugate roots.

24.14 Second Order Linear Equations, Another Take

There is another way to approach the problem of solving a second order linear equation. Before examining the solutions to equation (24.12.4), we'll take a brief look at this other method of solution.

It can be converted into a linear system. Define

$$z = \dot{y}.$$

Then $\dot{z} = \ddot{y}$, enabling us to rewrite equation (24.12.4) as

$$a\dot{z} + bz + cy = 0 \quad \text{or} \quad \dot{z} = -\frac{b}{a}z - \frac{c}{a}y$$

We can combine these two equations in matrix form:

$$\frac{d}{dt}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}.$$

Setting

$$\mathbf{A} = -\frac{1}{a}\begin{pmatrix} 0 & -a \\ c & b \end{pmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{pmatrix} y \\ z \end{pmatrix}$$

we can write the second order equation as a first order matrix equation

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}.$$

The characteristic polynomial of $\mathbf{A}$ is now

$$\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0 \quad \text{or} \quad a\lambda^2 + b\lambda + c = 0$$

which is the same as equation (24.13.5). This means that the left-hand side of equation (24.13.5) really is a characteristic polynomial.

If initial values of y and $\dot{y} = z$ are given, the solutions to this vector equation are

$$\boldsymbol{x}(t) = \exp(t\mathbf{A})\,\boldsymbol{x}_0 = \exp(t\mathbf{A})\begin{pmatrix} y_0 \\ z_0 \end{pmatrix}.$$

Notice that with two variables, we need initial data for each. This also happens in k^{th} order equations where the initial value problem requires initial data for y and and its derivatives, up through the $(k-1)^{st}$ derivative.

24.15 Case 1: Distinct Real Roots

The Principle of Superposition tells us that any linear combination of solutions to our second order equation is also a solution.

This applies when we have distinct real roots, r_1 and r_2. Not only are $e^{r_1 t}$ and $e^{r_2 t}$ solutions, but so is any linear combination

$$y(t) = k_1 e^{r_1 t} + k_2 e^{r_2 t}. \tag{24.15.6}$$

In fact, this is a general solution, meaning that with appropriate choice of coefficients k_1 and k_2, it will solves any initial value problem with $y(0) = y_0$ and $\dot{y}(0) = y_1$.

Here $\dot{y} = r_1 k_1 e^{r_1 t} + r_2 k_2 e^{r_2 t}$, so we must solve the linear system

$$\begin{aligned} y_0 &= y(0) = k_1 + k_2 \\ y_1 &= \dot{y}(0) = r_1 k_1 + r_2 k_2. \end{aligned}$$

In matrix form, this is

$$\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ r_1 & r_2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}. \tag{24.15.7}$$

The matrix has determinant $r_2 - r_1 \neq 0$ because the roots are distinct. It follows that there are unique k_1 and k_2 that solve the initial value problem. The formula for y in equation (24.15.6) really is a **general** solution.

In fact, if we had initial data at time t_0, the same procedure would apply, except that equation (24.15.7) would determine $k_i e^{r_i t_0}$, from which k_1 and k_2 can be calculated.

Notice that if $r_1 = r_2$ as in case 2, the determinant of the matrix is zero and we have a problem, a big problem. It needs special handling, and Case 2 addresses this.

24.16 Case 2: Repeated Real Roots

For insight, we turn to the matrix representation. When repeated real roots have only a single eigenvector, we use the Jordan form

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

We need to take the exponential of t times this, which is

$$\begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix},$$

where $te^{\lambda t}$ appears on the superdiagonal because we took the λ-derivative of the diagonal term.

This says that the solution will be a linear combination of $e^{\lambda t}$ and $te^{\lambda t}$ when λ is a root with algebraic multiplicity two and geometric multiplicity one.

Let's try that in our second order equation. Here there is only one root r, and $r = -b/2a$. Now

$$\begin{aligned} \frac{d}{dt}(te^{rt}) &= e^{rt} + tre^{rt} \\ \frac{d^2}{dt^2}(te^{rt}) &= re^{rt} + re^{rt} + r^2 e^{rt} \\ &= 2re^{rt} + r^2 e^{rt} \end{aligned}$$

Substituting $y = te^{rt}$ into our equation (24.12.4), we obtain

$$a\ddot{y} + b\dot{y} + cy = (ar^2 + br + c)te^{rt} + (2ar + b)e^{rt} = (2ar + b)e^{rt} = 0$$

because $r = -b/2a$.

The general solution is

$$y(t) = k_1 e^{rt} + k_2 te^{rt}.$$

This can be verified by showing that it can satisfy any initial conditions. Here $y(0) = k_1$ and $\dot{y}(0) = rk_1 + k_2$, a system that can be solved by substitution.

24.17 Case 3: Complex Conjugate Roots

This brings us to the case of complex roots. As long as a, b, and c, are all real numbers, the complex roots will be conjugates. We write the roots as $\alpha + \beta i$ and $\alpha - \beta i$ with α, β real. The general solution should have the form

$$y(t) = k_1 e^{(\alpha+\beta i)t} + k_2 e^{(\alpha-\beta i)t}.$$

The key to understanding the solution here is *Euler's Formula*. For any real x,[2]

$$e^{ix} = \cos x + i \sin x$$

It follows that

$$e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos \beta t + i \sin \beta t).$$

Notice that $\overline{e^{(\alpha+\beta i)t}} = e^{(\alpha-\beta i)t}$.

Our proposed general solution is

$$y(t) = k_1 e^{(\alpha+\beta i)t} + k_2 e^{(\alpha-\beta i)t}.$$

We may have to let the coefficients k_i be complex in order to get a real solution. In fact,

$$\overline{y(t)} = \overline{k_1} e^{(\alpha-\beta i)t} + \overline{k_2} e^{(\alpha+\beta i)t}$$

so the solution is real, $y(t) = \overline{y(t)}$, if and only if $k_1 = \overline{k_2}$.

We can then rewrite

$$\begin{aligned} y(t) &= k_1 e^{(\alpha+\beta i)t} + \overline{k_1} e^{(\alpha-\beta i)t} \\ &= e^{\alpha t}\left[k_1 e^{i\beta t} + \overline{k_1}\,\overline{e^{i\beta t}}\right] \\ &= 2e^{\alpha t}\operatorname{Re}\left(k_1 e^{i\beta t}\right) \end{aligned}$$

where Re denotes the real part of a complex number.

Since $k_1 e^{\beta} it$ is a linear combination of $\cos \beta t$ and $\sin \beta t$, so is its real part. It follows that the general form of a real-valued solution y can be written

$$y(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

when

$$\dot{y}(t) = \alpha y(t) + e^{\alpha t}\big(-\beta c_1 \sin \beta t + \beta c_2 \cos \beta t\big).$$

Here $y(0) = c_1$ and $\dot{y}(0) = \alpha c_1 + \beta c_2$, making it easy to match any initial data at time zero.

[2] There are several ways to derive Euler's formula. One method is to use power series and rewrite the power series for e^{ix} in terms of the power series for the sine and cosine.

24.18 Inhomogeneous Second Order Equations

A variation on the homogeneous equation (24.12.4) is an inhomogeneous equation of the form

$$a\ddot{y} + b\dot{y} + cy = g(t) \tag{24.18.8}$$

with $a, b, c \in \mathbb{R}$. The term $g(t)$ is called a *forcing term*. Such equations can represent a system where some external force is applied, as when a radio receives an external signal. In the case of a radio, the effects are generally unremarkable, unless the radio is tuned to the signal's frequency.

Solutions to equation (24.18.8) can be written as a *particular solution*, y_p, which solves the inhomogeneous equation plus a general solution, y_g for the associated homogeneous equation.

Theorem 24.18.1. *Suppose $y_p(t)$ solves the inhomogeneous equation (24.18.8). Then if $y_g(t)$ is a solution to the homogeneous equation (24.12.4), $y_g(t) + y_p(t)$ is also a solution to the inhomogeneous equation (24.18.8).*

Moreover, if $y(t)$ is any other solution to the inhomogeneous equation (24.18.8), then $y(t) - y_p(t)$ is a solution to the homogeneous equation (24.12.4)

Finally, if y_g is a general solution to the homogeneous equation, $y_p + y_g$ is a general solution to the inhomogeneous equation.

Proof. Let $\mathbf{T}(y) = a\ddot{y} + b\dot{y} + cy$. Then $\mathbf{T}$ is linear in y. Now suppose y_g obeys $\mathbf{T}(y_g) = 0$. Then $\mathbf{T}(y_p + y_g) = \mathbf{T}(y_p) + \mathbf{T}(y_g) = g(t) + 0 = g(t)$, so $y_g + y_p$ solves (24.18.8).

If both y and y_p solve (24.18.8), then $\mathbf{T}(y - y_p) = \mathbf{T}(y) - \mathbf{T}(y_p) = g(t) - g(t) = 0$, showing that $y - y_p$ solves the homogeneous equation $\mathbf{T}(y) = 0$.

For the last part, suppose the $y_p(0) = y_{p0}$ and $\dot{y}_p(0) = y_{p1}$. To find a solution y to the inhomogeneous system with initial data $y(0) = y_0$ and $\dot{y}(0) = y_1$, it suffices to adjust the coefficients of y_g, which solves the homogeneous system so that $y_g(0) = y_0 - y_{p0}$ and $\dot{y}_g(0) = y_1 - y_{p1}$. Since y_g is a general solution, this is always possible. ∎

24.19 Inhomogeneous Case: Undetermined Coefficients

Theorem 24.18.1 points to the importance of particular solutions for inhomogeneous equations. How do we find them?

One method of finding a particular solution is the *method of undetermined coefficients*.[3] We look for a particular solution based on the forcing term and its derivatives.

For example, suppose

$$\ddot{y} + y = t^2.$$

We try a linear combination of t^2 and its derivatives, $2t$ and 2.

$$y_p(t) = At^2 + Bt + C$$

Then

$$\begin{aligned} \dot{y}_p &= 2At + B \\ \ddot{y}_p &= 2A \end{aligned}$$

so

$$\ddot{y}_p + y = 2A + At^2 + Bt + C = t^2.$$

Since t^2, t, and 1 are linearly independent functions, we match the coefficients: $A = 1$, $B = 0$, and $2A + C = 0$. Then $C = -2$ and our particular solution is

$$y_p(t) = t^2 - 2.$$

Since the homogeneous system has general solution $k_1 \cos t + k_2 \sin t$, a general solution to the inhomogeneous system is

$$y(t) = k_1 \cos t + k_2 \sin t + t^2 - 2.$$

[3] This is not to be confused with a more advanced technique, variation of parameters. We will not consider that in this course.

24.20 More Undetermined Coefficients

Let's try another one. Here

$$\ddot{y} - 5\dot{y} + 6y = e^t. \qquad (24.20.9)$$

Here e^t is its own derivative. We try $y_p = Ae^t$. Then

$$Ae^t - 5Ae^t + 6Ae^t = e^t \quad \text{so} \quad 2A = 1.$$

The particular solution is $y_p = e^t/2$ and the general solution is

$$y(t) = k_1 e^{2t} + k_2 e^{3t} + \frac{1}{2}e^t.$$

24.21 Resonances I

If we alter equation (24.20.9) slightly, a strange thing happens. Consider

$$\ddot{y} - 5\dot{y} + 6y = e^{2t}. \tag{24.21.10}$$

If we try the method of undetermined parameters, it doesn't work. With $y = Ae^{2t}$, we obtain

$$\ddot{y} - 5\dot{y} + 6y = 4Ae^{2t} - 10Ae^{2t} + 6Ae^{2t} = 0.$$

There is no way we can make zero equal e^{2t}.

We previously saw that if the root 2 were repeated, we would have solutions of the form te^{2t}. Such a function survives one application of $\mathbf{T}(y) = \dot{y} - 2y$, but not two. That is

$$\mathbf{T}(te^{2t}) = e^{2t} + 2te^{2t} - 2te^{2t} = e^{2t},$$

but if we apply $\mathbf{T}$ a second time,

$$\mathbf{T}\big(\mathbf{T}(te^{2t})\big) = \mathbf{T}(e^{2t}) = 2e^{2t} - 2e^{2t} = 0,$$

we get zero. That is what we need here. We need to have a leftover term involving e^{2t}.

So we try $y = Ate^{2t}$. Then

$$\begin{aligned}\dot{y} &= Ae^{2t} + 2Ate^{2t}\\ \ddot{y} &= 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t}\\ &= 4Ae^{2t} + 4Ate^{2t}.\end{aligned}$$

We substitute in equation (24.21.10) to find

$$\begin{aligned}\ddot{y} - 5\dot{y} + 6y &= 4Ae^{2t} + 4Ate^{2t} - 5Ae^{2t} - 10Ate^{2t} + 6Ae^{2t} + 6Ate^{2t}\\ &= 5Ae^{2t}\\ &= e^{2t},\end{aligned}$$

so $A = 1/5$. Our attempt worked.

Our particular solution is $y_p = te^{2t}/5$. You'll notice that the particular solution grows faster than the corresponding piece of the general solution. This is an example of a *resonance*.

24.22 Resonances II

In some cases resonances can be more dramatic. Consider

$$\ddot{y} + y = \cos t.$$

Here the general solution to the homogeneous system is

$$y_g(t) = k_1 \cos t + k_2 \sin t$$

and there is no point in trying either $\sin t$ or $\cos t$ as particular solutions. We must multiply by t. We try

$$y_p(t) = At \cos t + Bt \sin t$$

when

$$\begin{aligned}
\dot{y}_p &= A \cos t - At \sin t + B \sin t + Bt \cos t \\
\ddot{y}_p &= -A \sin t - A \sin t - At \cos t + B \cos t + B \cos t - Bt \sin t \\
&= -2A \sin t - At \cos t + 2B \cos t - Bt \sin t.
\end{aligned}$$

It follows that

$$\ddot{y}_p + y_p = -2A \sin t + 2B \cos t = \cos t$$

so $A = 0$ and $B = 1/2$. The general solution is

$$y_g(t) = \frac{1}{2} t \cos t + k_1 \cos t + k_2 \sin t.$$

Notice how the general solution is dominated by the $(t/2) \cos t$ term as t gets large. The system amplifies the signal forcing it.

In real-world radios and cellphones the amplification is not infinite, but can still be quite substantial.

24.23 Existence of Solutions: Peano Existence Theorem

We have blithely attempted to solve differential equations without really having any idea whether they actually have solutions. We will put an end to that by giving some general theorems on existence and uniqueness. The basic existence theorem is the Peano (or Cauchy-Peano) existence theorem.

Peano Existence Theorem. *Let* $U \subset \mathbb{R}^2$ *be open and* $F\colon U \to \mathbb{R}$ *be continuous. Then the initial value problem*

$$\begin{aligned} \dot{y}(t) &= F(t, y) \\ y(t_0) &= y_0 \end{aligned}$$

has a solution y *defined on a neighborhood of* t_0.

It's possible to prove that differential equations have solutions under weaker conditions. If you need such, you might look up the Carathéodory existence theorem.

24.24 Uniqueness of Solutions: Picard-Lindelöf Theorem

In general, solutions to differential equations need not be unique. Showing that the solution is unique requires stronger assumptions, as those given in the Picard-Lindelöf theorem.

The key assumption involves a special type of continuity.

Lipschitz Continuity. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *Lipschitz continuous* if there is a $K > 0$ with $|f(x) - f(x')| \le K|x - x'|$ for all $x, x' \in \mathbb{R}$.

One consequence of Lipschitz continuity is that if f is both differentiable and Lipschitz continuous, its derivative is bounded by K, $|f'(x)| \le K$. Conversely, if the derivative is bounded, the Mean Value Theorem can be used to show that the function is Lipschitz continuous with the same bound.

Picard-Lindelöf Theorem. *Let $U \subset \mathbb{R}^2$ be open and $F\colon U \to \mathbb{R}$ be continuous. If there is a $K > 0$ with $|F(t,y) - F(t,y')| \le K|y - y'|$ for all $(t,y), (t,y') \in U$, then there is $\varepsilon > 0$ so that the initial value problem*

$$\begin{aligned} \dot{y}(t) &= F(t,y) \\ y(t_0) &= y_0 \end{aligned}$$

has a unique solution on the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$.

There are also multidimensional versions of the Peano and Picard-Lindelöf Theorems.

24.25 A Solution that is Not Unique

Consider the differential equation

$$\dot{y} = +|y|^{1/2}$$

It is clear that $y(0) = 0$ is a solution. Another solution with $y(0) = 0$ is

$$y(t) = \begin{cases} 0 & \text{when } t \leq 0 \\ t^2/4. & \text{when } t > 0 \end{cases} \tag{24.25.11}$$

So there are at least two solutions obeying $y(0) = 0$.

In fact, there are infinitely many. The other solutions follow $y(t) = 0$ until a time t_0, when it switches to $y(t) = (t - t_0)^2/4$. Several such solutions are illustrated in Figure 24.25.1.

In this case, $|y|^{1/2}$ is not Lipschitz continuous, so the Picard-Lindelöf theorem does not guarantee uniqueness. The Peano theorem applies because $|y|^{1/2}$ is continuous, but it doesn't say anything about uniqueness, only existence.

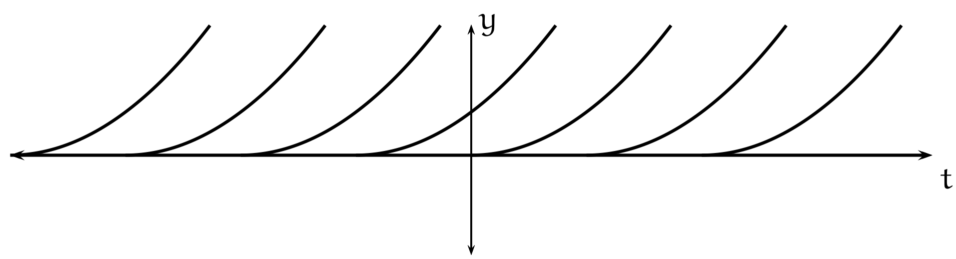

Figure 24.25.1: The diagram shows how profoundly non-unique the solutions to equation (24.25.11) are. One solution is the steady state $y(t) = 0$. At any point t, another solution branches off. Seven of them are illustrated on the diagram.

24.26 Long-run Behavior: Phase Portraits 12/01/20

We close this chapter with a quick look at long-run behavior. One way to study the long-run behavior of an autonomous differential equation is to use a phase portrait. The phase portrait shows, for any value of y, whether $\dot{y}$ is increasing or decreasing.

Let's start with an autonomous ordinary differential equation

$$\dot{y} = F(y)$$

To construct the phase portrait, we draw the graph of F. Then $\dot{y}$ is positive when the graph is above the axis and negative when the graph is below the axis. We indicate the direction of motion by arrows on the axis. When the graph is on the axis, $\dot{y} = 0$, indicating a steady state.

Let's see how this works when $F(y) = y(2 - y)$. There are two steady states, where $F(y) = 0$. These occur at $y = 0$ and $y = 2$.

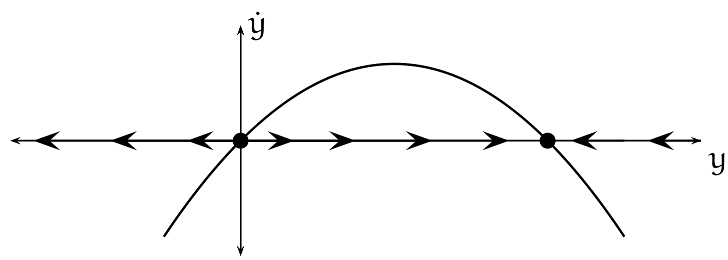

Figure 24.26.1: There are two steady states, at $y = 0$ and $y = 2$. The big arrows indicate that the steady state at $y = 0$ is unstable. If we start a little to the left, we move away to the left, and starting a little to the right means we move away to the right.

The steady state at $y = 2$ is asymptotically stable. If we start a bit to the left, we move back toward the steady state at 2. If we start to the right, we also move back toward it.

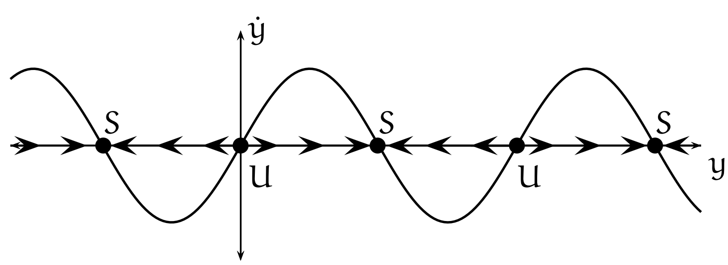

Figure 24.26.2: This phase portrait is for the equation $\dot{y} = \sin y$. Five steady states are indicated, three are asymptotically stable (S) and two are unstable (U).

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