Micro II Midterm, October 14, 2014

1. Does $u(x, y, z) = (1 + x)(1 + y) + y^{1/2} + z$ have an equivalent additive separable representation?

   Answer: We check whether preferences are completely separable. The function u is obviously increasing in each argument, so we check marginal rates of substitution of each pair. Now $MRS_{xz} = (1 + y)/1 = 1 + y$ depends on y, so u does not induce an order on $\{1, 3\}$. This implies it is not completely separable, and so not representable by an additive separable utility function.

2. Suppose there are 3 states, $s = 1, 2, 3$. Define lotteries $L_1 = (1/2, 1/2, 0)$, $L_2 = (0, 1/3, 2/3)$ and $L_3 = (2/3, 0, 1/3)$. Suppose a consumer has von Neumann-Morgenstern preferences with $u(L_1) = 0$, $u(L_2) = 1$ and $u(L_3) = 3$. Find the utility of state 1.

   Answer: As vectors, we solve the following for $\mathbf{x}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \mathbf{x}.$$

   That yields $\mathbf{x} = (1/5, -3/5, 6/5)^T$. Let $L = (1, 0, 0)^T$. As vectors, $L = L_1/5 - 3L_2/5 + 6L_3/5$ We rewrite as $\frac{5}{8}L \oplus \frac{3}{8}L_2 = \frac{2}{8}L_1 \oplus \frac{6}{8}L_3$. It follows that $5u(L) + 3u(L_2) = 2u(L_1) + 6u(L_3) = 18$, so $5u(L) = 18 - 3u(L_3) = 15$ and $u(L) = 5$.

3. There are two goods. Suppose a consumer has the utility function $u(x_1, x_2) = x_1 + 2x_2$ and consumption set $\mathfrak{X} = \mathbb{R}^2_+$. Let prices be $\mathbf{p} = (1, p) \gg 0$ and income be $m > 0$.

   a) Find the ordinary (Marshallian) demand $\mathbf{x}(\mathbf{p}, m)$.

   Answer: When $p = 2$, the budget constraint coincides with the indifference curve and every point on the line is a maximizer. When $p < 2$, a util of good 1 costs \$1 while a util of good 2 costs less than \$1. The consumer buys only good 2. If $p > 2$, the opposite is true and the consumer buys only good 1. Summing up,

$$\mathbf{x}((1, p), m) = \begin{cases} (m, 0) & \text{if } p > 2 \\ \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + 2x_2 = m\} & \text{if } p = 2 \\ (0, m/p) & \text{if } p < 2. \end{cases}$$

   b) Compute the indirect utility function $v(\mathbf{p}, m)$.

   Answer: Indirect utility is $v(p, m) = u(\mathbf{x}(p, m)) = \max\{m, 2m/p\}$.

   c) Find the expenditure function $e(\mathbf{p}, \bar{u})$.

   Answer: Use duality, $\bar{u} = v(\mathbf{p}, m) = \max\{e(\mathbf{p}, \bar{u}), 2e(\mathbf{p}, \bar{u})/p\}$. If $p < 2$, $\bar{u} = 2e/p$ so $e(\mathbf{p}, \bar{u}) = p\bar{u}/2$. If $p > 2$, $\bar{u} = e(\mathbf{p}, \bar{u})$. This can be summed up as $e((1, p), \bar{u}) = \bar{u}\min\{1, 2/p\}$.

4. In $\mathbb{R}^3$, consider the linear activity model with technology set Y generated by

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

a) Show that Y obeys the no free lunch condition.

Answer: Suppose there is a non-negative vector $\mathbf{y}$ in Y. Then there are non-negative z_i with

$$z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + z_3\mathbf{a}_3 = \begin{pmatrix} z_1 + z_2 - z_3 \\ -z_2 \\ -z_1 + z_3 \end{pmatrix} \geq \mathbf{y} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $z_2 \geq 0$ and $-z_2 \geq 0$, $z_2 = 0$. That leaves $z_1 \geq z_3$ and $z_3 \geq z_1$, so $z_1 = z_3$. Putting the first and last together we find $z_1 \geq z_3 \geq z_1$, implying $z_1 = z_3$ and $z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + z_3\mathbf{a}_3 = \mathbf{0}$, showing $\mathbf{y} \leq \mathbf{0}$. This establishes the no-free lunch condition.

b) For what prices is profit maximization possible (i.e., what is Y^*)?

Answer: Since the technology is constant returns to scale, the maximum possible profit is zero. Moreover, the polar coincides with the dual cone. Thus profit can only be maximized when the price vector is in the dual cone Y^*. We saw in class that $Y^* = \{\mathbf{p} \in \mathbb{R}^3 + : \mathbf{p} \cdot \mathbf{a}_i \leq 0$ for every $i\}$.