Micro II Final, December 10, 2015

1. Suppose a consumer has utility $\sum_{t=0}^{\infty} \delta^t u(c_t)$ where $0 < \delta < 1$ and the felicity function is $u(c) = \ln c$. Prices are $p_0 = 2$ and $p_t = (1+r)^{-t}$ where $r > 0$ is the interest rate. The budget constraint is $\sum_{t=0}^{\infty} p_t c_t = 123$. Let $r = 0.05$ and $\delta = 1/(1+0.05)$. Solve the intertemporal consumer's problem.

   Answer: The first-order conditions are
$$\frac{\delta c_t}{c_{t+1}} = \frac{\delta u'(c_{t+1})}{u'(c_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{1+r}.$$
   Plugging in the values, we find $c_t = c_{t+1}$ for $t = 1, 2, \ldots$ and $c_1 = 2(1+r)\delta c_0 = 3c_0$. Thus $c_t = 2c_0$ for $t = 1, 2, \ldots$.

   The budget constraint is then $123 = c_0 + 2c_0 \sum_{t=1}^{\infty}(1+r)^{-t} = c_0(1 + 2/r)$, so $c_0 = 3$. The optimal path is $(3, 6, 6, 6, \ldots)$.

2. Let $\mathfrak{X} = \mathbb{R}^2_+$. The indirect utility function is $v(\mathbf{p}, m) = (m - p_1 a_1 - p_2 a_2)/2\sqrt{p_1 p_2}$ for a_i given.

   a) Find the expenditure function.

   b) Find the utility function.

   Answer:

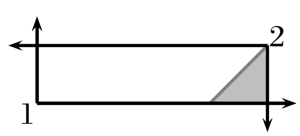
   a) By duality, $\bar{u} = v\big(\mathbf{p}, e(\mathbf{p}, \bar{u})\big) = \big(e(\mathbf{p}, \bar{u}) - \mathbf{p}\cdot\mathbf{a}\big)/2\sqrt{p_1 p_2}$. Solving for e, we obtain $e(\mathbf{p}, \bar{u}) = \mathbf{p}\cdot\mathbf{a} + 2\bar{u}\sqrt{p_1 p_2}$.

   b) We compute the Hicksian demands $\mathbf{h} = \mathbf{d}_\mathbf{p} e = (a_1 + \bar{u}\sqrt{p_2/p_1}, a_2 + \bar{u}\sqrt{p_1/p_2}$. It follows that $(h_1 - a_1)(h_2 - a_2) = \bar{u}^2$, so the utility function is $u(x_1, x_2) = \sqrt{(x_1 - a_1)(x_2 - a_2)}$.

3. Suppose $u_1(\mathbf{x}^1) = \max\{x_1^1, x_2^1\}$ and $u_2(\mathbf{x}^2) = \min\{x_1^2, x_2^2\}$, with endowments $\boldsymbol{\omega}^1 = (1, 1)$ and $\boldsymbol{\omega}^2 = (3, 0)$. Find the core. Be careful, u_1 is not a typo!

   Answer: We start by finding the Pareto set. Note that $u_2 \le \omega_2 = 1$. If $x_1^2 > x_2^2$, we can Pareto improve by giving the excess of good 1 to consumer 1 who will then have $u_1 = 4 - x_2^2$. Thus $x_1^2 \le x_2^2$ at any Pareto optimum. Once we have given at least $4 - x_2^2$ of good 1 to consumer 1, consumer 1's utility cannot be further increased. It follows that the Pareto optima are $\{\big((4-x, 1-y), (x, y)\big) : 0 \le y \le 1, x \le y\}$ yielding utility $u_1 = 4 - x$ and $u_2 = x$. (This is the same as problem 11.4.)

   Individual rationality requires $u_1 = 4 - x \ge 1$ and $u_2 \ge 0$, both of which are satisfied at all the Pareto optima.



4. Suppose an exchange economy has 2 consumers and 2 goods. Consumer one has endowment $\boldsymbol{\omega}^1 = (1, 0)$. Utility is $u_1(\mathbf{x}^1) = \sqrt{x_1^1}$. Consumer two has endowment $\boldsymbol{\omega}^2 = (0, 1)$. Utility is $u_2(\mathbf{x}^2) = \sqrt{x_1^2} + \sqrt{x_2^2}$. The consumption sets are $\mathfrak{X}_i = \mathbb{R}^2_+$. Show that there is no competitive equilibrium.

   Answer: If $p_1 = 0$, consumer one's demand for good one is infinite. This cannot be an equilibrium.

   Consider the case $p_1 > 0$, demand by consumer one is $\mathbf{x}^1(\mathbf{p}) = (1, 0)$. Demand by consumer two is
$$\mathbf{x}^2(\mathbf{p}) = \left(\frac{p_2}{p_1(p_1 + p_2)}, \frac{p_1}{p_2(p_1 + p_2)}\right).$$
   In equilibrium, $\mathbf{x}(\mathbf{p}) \le \boldsymbol{\omega} = (1, 1)$. If we normalize prices so that $p_1 + p_2 = 1$, $\mathbf{x}(\mathbf{p}) = (1 + p_2/p_1, p_1/p_2)$. Equilibrium in market 1 requires $p_2 = 0$, which means demand for good 2 is infinite. Equilibrium in market two requires $p_1/p_2 = 1$, in which case there is excess demand for good 1. There is no equilibrium.

5. An economy has two goods and two identical Cobb-Douglas consumers with $u_i(\mathbf{x}^i) = \sqrt{x_1^i x_2^i}$. The total endowment is $(0, 6)$. There is one constant returns to scale firm that produces good 1 and uses good 2 as its only input. The production function is $f(z) = 2z$.

   Find all Pareto optimal allocations of goods and the corresponding net output vector.

   Answer: The net output vector will have the form $\mathbf{y} = (-2y_2, y_2)$ with $y_2 \le 0$. Here the marginal rate of transformation is $\text{MRT}_{12} = 1/2$. This must also be the marginal rate of substitution at any interior Pareto optimum. Note that since utility is zero if there is no production, the production technology must be used.

   Now $\text{MRS}^i_{12} = x_2^i/x_1^i = 1/2$, so $2x_2^i = x_1^i$. Summing over both consumers, aggregate consumtion obeys $2x_2 = x_1$. Since good 1 can only be obtained from the production sector, $x_1 = -2y_2$ and $x_2 = 6 + y_2$. Thus $x_2 = -y_2$ and $x_2 = 6 + y_2$. It follows that $x_2 = 3$ and $y_2 = -3$, so $x_1 = 6$.

   Since both consumers will consume in the same proportions, $\mathbf{x}_1 = \alpha(6, 3)$ and $\mathbf{x}_2 = (1 - \alpha)(6, 3)$ for some α between 0 and 1. Also, $\mathbf{y} = (6, -3)$. These are the Pareto optimal allocations.